Ex Ante Valuation of Systemic Risk: A Financial Equilibrium Networks Approach*

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Abstract

How should regulators price deposit insurance in the face of potential contagion? We answer this question in the context of a tractable network model of the interbank market. The model features claims of differing priorities and endogenous fire sales. We derive expressions for risk premia and isolate the impact of network risk. We show that the premia increase non-linearly in network risk. We find that network risk magnifies the gap between the premia of debt and equity. We also uncover a novel trade-off between aggregate losses and those borne by depositors in the event of lengthy cascades of defaults.

Keywords: Networks; Systemic Risk Premia; Deposit Insurance
JEL codes: G18, G21, G32, G33, L14

1 Introduction

All depositors of [Silicon Valley Bank] will be made whole. No losses associated with the resolution of Silicon Valley Bank will be borne by taxpayers. Shareholders and certain unsecured debt holders will not be protected.
(Federal Deposit Insurance Corporation, Monday March 13, 2023)

Assessing the risks hidden in bank balance sheets is imperative. These risks affect account holders with balances in excess of the maxima guaranteed by deposit insurance

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and in turn, the regulators that set the corresponding policy instruments. Riskiness of a given bank affects other financial institutions, from which they have been extended credit, as well as the shareholders with a stake in its profits and central bankers that seek financial stability. This latter group relies on stress tests, which assess the likelihood of financial institutions surviving a given scenario for losses on the asset side of their balance sheets. However, stress tests are inadequate when it comes to the evaluation of *ex ante* risks. *Ex ante* assessments are essential in some contexts, such as the setting of insurance premia for deposits. In this setting, regulators must consider the joint probability distribution of all possible shocks, which may impact the balance sheets of banks. An *ex ante* risk assessment is an arduous undertaking, by virtue of the connectivity of banks between each other and additional financial institutions, creating inter-dependence of their future solvency statuses. As observed by Jackson and Pernoud (2021), any reliable risk assessment must account for the whole network structure of such financial inter-dependencies.

With full knowledge of the structure of the network of financial obligations, *ex ante* risk computation remains an extremely challenging problem.[1] These types of risk valuations require (a) knowledge of the joint distribution of asset value losses across the financial system and (b) a clearing process for each possible realisation of that distribution. A vast literature, stemming from Eisenberg and Noe (2001), developed algorithms for clearing a system subsequent to some initial losses, where all residual banks at the process' cessation are solvent. *Ex ante* assessments are complicated from the perspective that such a clearing algorithm would need to be repeated for all possible realisations of initial losses. Moreover, the burden of these computations can be further compounded by uncertainty about the structure of the network.

In this paper, we ask the question of how the inter-connectedness of the financial system affects *ex ante* risk evaluations. We develop a simple model, which retains tractability in the face of a demanding combinatorial problem, while also capturing essential characteristics of real financial networks. We show that risk premia are reflective of three different types of risk — idiosyncratic risk, counter-party risk and network risk. Idiosyncratic risk emanates from outside the financial network, embodying the risk associated with banks' investment activities, such as mortgages or business loans. Banks also face risk that their counter-parties default on their financial obligations, either directly due to their own outside investments, or indirectly from failing banks further up the chain. We refer to the former as counter-party risk and the latter as network risk. This distinction makes evident that information about one's own counter-parties, while ignoring the remainder of the network structure, is insufficient to account for all possible scenarios driving bank financial distress.

[1]Although financial institutions have a number of instruments at their disposal to insure against risks (e.g. credit defaults swaps and other derivatives), for various reasons, including that the providers went bust, these proved inadequate during the 2009 Global Financial Crisis. To keep things simple, we assume that such insurance opportunities are not available. For example, this could be because it is costly to verify bank returns; this would also explain why the outside liabilities of banks are debt contracts and verification is only triggered when the bank becomes insolvent (see, for example, Townsend, 1979). Moreover, Zawadowski (2013) argues that the pricing of such insturments does not include externalities arising due to network effects.

Our model features two essential ingredients for addressing this research question. The first is claims of differing priority statuses. It is clear that claims of higher priority face lower *ex ante* risk, which is reflected in their corresponding risk premia. However, an important point our analysis highlights, is that network risk widens the gap between the risk premia of equity and those of claims with higher priority. In so doing, we make a first contribution in offering network risk as a novel explanation for the equity premium puzzle. We make a second contribution in formalising the relationship between network risk and the appropriate risk premia for deposit insurance. When a bank extends a loan, it imposes a negative externality on the depositors of other institutions. Should the borrower default, the bank may be unable to honour its own liabilities, putting depositors of banks higher up the network at risk.

A second crucial ingredient in our model is endogenous fire sales. When a bank has insufficient liquidity to meet its obligations, it may be able to remain solvent through selling its illiquid assets. It follows that its survival depends on the market value of these assets. During crisis times, there may be many banks attempting to offload these assets, in the face of fewer buyers, as banks' priorities move towards consolidation rather than expansion. When the liquidation value of a bank's assets is too low, it will default, giving rise to the possibility of distress cascading to counter-parties through the links of the network. Our model features a liquidation value function that depends on the quantity of illiquid assets marketed for sale. Accounting for this process can lead to considerable amplification of the shocks, which caused the initial failures. We make a third contribution in being the first study to bring endogenous fire sales to a model of *ex ante* risk assessment. Omitting this feature downplays the degree of network risk in the inter-bank system, under-stating the appropriate size of deposit insurance premia. This potentially places excessive pressure on taxpayer funds, when it comes to financing such a scheme.

We leverage our framework to understand the risk implications of uncertainty regarding the financial network's structure. Real networks are not fixed — new links can be formed or the weight of an existent link may change. These changes can carry profound implications for both aggregate losses and the distribution of losses across the network. Our fourth contribution is a counter-intuitive finding that, across network architectures, a novel trade-off exists between aggregate losses and those incurred by priority stakeholders. In the context of a cycle network (e.g. a circle of one-way directed links), chains with greater length potentially lead to longer cascades of defaults. While a shorter chain can limit the extent of contagion, the sequence of defaults can complete a revolution around the cycle, eroding cash-flow leftover for distribution to priority claimants. As such, uncertainty about network structure brings important implications for the risk premia of different liability classes.

Our starting point is the model in Caballero and Simsek (2013), where all banks are symmetric and located on a single directed cycle (each bank obtains a loan from exactly one bank and offers a loan to exactly one bank). We allow for arbitrary distributions of multiple shocks across the banking system and derive the corresponding risk premia on deposits, bank loans and equity. This simple model clarifies the complications for *ex ante* risk assessment arising from the introduction of priority claims and endogenous fire sales.

Then, we consider two alternative network structures that capture important features of real financial networks; namely, multiple directed cycles and hierarchies.[2]

As Jackson and Pernoud (2021) illustrate, the presence of directed cycles in the financial network gives rise to the possibility of multiple equilibria. Moreover, as we noted above, when there are bank liabilities with priority status, the lengths of these cycles are crucial for the distribution of losses across claimholders. Motivated by this, we consider networks composed by asymmetric cycles, whose length distributions are arbitrary. As others have shown, aggregate losses depend on both the distribution of shocks across the system and the structure of the network. Our work is the first to show how uncertainties regarding (a) a bank's position in the network and (b) the structure of the network affect *ex ante* risk assessments.

Another important feature of real financial networks is hierarchies. There is considerable heterogeneity in real interbank networks in terms of number of links and more specifically in the number of creditors and debtors. In order to understand how hierarchies affect *ex ante* risk assessments we also analyze a *complete order*, which is a network structure that is strictly hierarchical. Our main finding for this structure is that the risks borne by various claimholders depend entirely of the position of their bank in the network. One implication of our analysis is that deposit risk premia should not be uniform across the banking system but should reflect the position of banks in the network.[3]

Literature Review[4] Our paper contributes to that part of the literature that views fire sales as an important amplification mechanism of exogenous shocks. In their review article on fire sales, Shleifer and Vishny (2011) emphasise that 'because of fire sales, risk becomes systemic', (p.30). Their insight is that in financial network models without fire sales, or other 'wedges' such as bankruptcy costs, aggregate losses are simply equal to those losses incurred by the initial shocks. In that case, the only thing left for the clearing process to determine is the number of institutions that will be liquidated, where the prices of assets to be sold are equal to their book values by assumption. However, evidence is mounting that fire sales played a prominent role in amplifying shocks during the Great Depression (Mitchener and Richardson, 2019) and during the 2009 Global Financial Crisis (Adrian and Shin, 2010; Brunnermeier, 2009; Gorton and Metrick, 2012). There is also evidence on the impact of fire sales in the manufacturing (Benmelech and Bergman, 2009) and in the housing sector (Campbell *et al.*, 2011). Much theoretical work on fire sales has been based on models that do not explicitly account for the connections between firms (Acharya and Shin, 2010; Diamond and Rajan, 2011; Greenwood *et al.*, 2015; Guerrieri and Shimer, 2014; Kurlat, 2021; Shleifer and Vishny, 1992).

[2]We have also considered directed cycles with a core-periphery setup. As long as the periphery mainly consists of small size financial institutions that borrow funds from the core, as in the U.K. financial system (Adams et al., 2010), our main results still apply.

[3]Our work formalises policy proposals put forward following the Global Financial Crisis of 2007/08 (see for example Saunders et al. (2009) and Strahan (2013)).

[4]The literature on contagion in financial networks is vast and there are already a number of review articles that provide good cover; Allen and Babus (2009), Bougheas and Kirman (2015), Glasserman and Payton (2016) and Jackson and Pernoud (2022).

Our work is more closely related to those papers that investigate how fire sales amplify shocks in networks. A vast literature has followed the work by Eisenberg and Noe (2001), by developing algorithms for clearing financial networks hit by some initial shocks, under a variety of suppositions about the transmission of shocks between institutions and the relationships between balance sheet entries. Many papers introduce liquidation costs, but assume that these costs, (like those of bankruptcy), are independent of the number of liquidated institutions (see, for example, Bardoscia *et al.*, 2015; Barucca *et al.*, 2018; Furfine, 2003; Rogers and Veraart, 2013). Our work belongs to that group of papers where liquidation costs (fire sales) depend on the supply of assets offered for liquidation.[5] We generalise the parsimonious setup of Caballero and Simsek (2013), allowing for sufficiently more complex network structures that still allow for the assessment of *ex ante* balance sheet risk when there exist liabilities with different priority rights. Amini *et al.* (2016), Cifuentes *et al.* (2005), Feinstein (2017) and Jackson and Pernoud (2022) develop clearing algorithms for networks with arbitrary structures and endogenous fire sales. However, none of these papers develop methods for *ex ante* balance sheet risk assessment and with the exception of Amini *et al.* (2016) do not allow for claims that have priority rights.[6]

Our work is also related to a small group of papers that develop interbank clearing mechanisms, similar to that in Eisenberg and Noe (2001), then use them to provide *ex ante* risk assessments. The models developed differ in the asset valuation functions they employ. In Veraart (2020) and in Elsinger *et al.* (2006), there is a common valuation function for the whole network. In Barucca *et al.* (2016), each bank does its own evaluation, but those valuations and the clearing algorithm ensures that the evaluations are consistent across banks. Glasserman and Young (2015) follow a different approach, by comparing an interbank network with a financial system consisting of the same number of banks but without any interbank obligations. They derive upper bounds for the probability of contagion and also for expected losses due to network effects. These papers allow for bankruptcy costs, which are independent of the number of liquidated institutions. Our study goes beyond this approach, by developing a model with endogenous liquidation costs, which is tractable yet sufficiently rich to allow for interesting comparative statics on the shock distribution and network structure. In our framework, each round of liquidations in the clearing algorithm has to go through another round of iterations after an adjustment is made for the change in liquidation values. We also cover new ground whereby introducing priority rights we are also able to provide separate *ex ante* risk assessments for outside debt, interbank obligations and equity.[7]

The remainder of this paper is organised as follows. Section 2 details the environment,

[5]There is another group of papers where liquidation costs are triggered by mark-to-market pricing of assets when institutions hold over-lapping portfolios (Caccioli *et al.*, 2014; Cont and Schaanning, 2017; Elliott *et al.*, 2014).

[6]Acemoglu *et al.* (2015) in an interbank network and Fisher (2014) in a cross-ownership model also allow for priority claims.

[7]There is some evidence from industrial data on the impact of network risk on stock returns. Ahern (2013) finds that industries that are more centrally located in the network of intersectoral trade earn relatively higher stock returns. The author argues that such industries are exposed to greater market risk because they have more vulnerable to sectoral shocks that cascade from one industry to another through trade links.

equilibrium and discusses risk for the baseline model with a single directed cycle. We use this baseline model to develop the tools that we will apply in the subsequent sections, to more complicated structures that capture real financial networks. Section 3 extends to arbitrary distributions of cycles. We will demonstrate how the presence of cycles in a network has important implications for the distribution of risk across those who hold bank liabilities. Moreover, we will identify the impact that changes in (a) the distribution of shocks across the network and (b) in the structure of the network have on *ex ante* risk evaluations. Section 4 considers the complete order network structure, which will allow us to study the impact of hierarchies on *ex ante* risk assessments. Section 5 concludes.

2 Single Cycle Network

2.1 Environment

We build-off the structure of Caballero and Simsek (2013) in considering a network comprised of a single directed cycle.

2.1.1 Architecture

There are $N \in \mathbb{Z}$ banks, denoted by b_i, with $i \in \{0, 1, ..., N-1\}$. Debt obligations flow in one direction where b_i has borrowed from b_{i+1} for $i \in \{0, 1, ..., N-2\}$ and b_{N-1} has borrowed from bank b_0. Figure 1 gives an graphical example of one such network with $N = 5$.

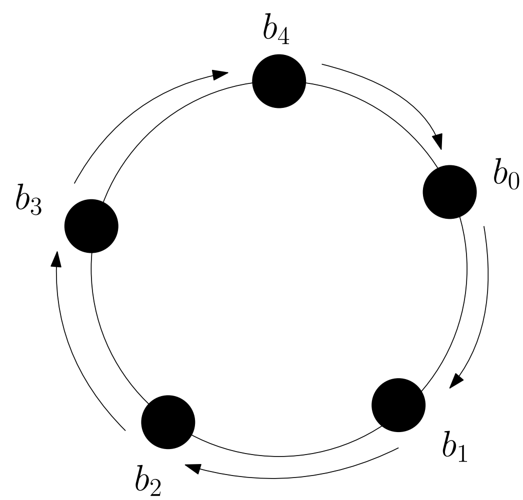

Figure 1: Directed cycle network where $N = 5$.

2.1.2 Balance Sheets

For the moment, we follow Caballero and Simsek (2013) and assume that all banks are symmetric. We will address issues related to network asymmetries in later sections. As such, we now focus on a representative balance sheet, after the realisation of shocks but before the settlement of any obligations to depositors and other banks. The status of the balance sheet will depend on whether the bank was hit by a shock. For a bank, which was not hit by a shock, the balance sheet is shown in Table 1 below.

Assets	**Liabilities**
Revenues (R)	Deposits (F)
Bank Loans (D)	Bank Deposits (D)
Non-liquid Assets (K)	Equity (E)

Table 1: Balance sheet.

The assets include the revenues from activities outside the network, R, loans offered to its debtor bank within the network, D, and other non-liquid assets, K. The liabilities include funding from depositors, F, the value of its obligation to its creditor bank in the network, D, and the value of its equity, E. Given that all banks are symmetric, the value of interbank debt, D, is the same on the two sides of the balance sheet. We assume that in the case of insolvency, depositors have a priority claim to the bank's assets. A bank that is not hit by a shock, which also has a solvent creditor bank, is itself solvent: $E = K + R - F > 0$. Initially, we introduce the following assumptions:[8]

Assumption 1: $R - F > 0$,

Assumption 2: $D - F > 0$.

Assumption 1 implies that a bank not hit by a shock has enough liquidity to cover its obligations to depositors. Assumption 2 implies that as long the bank loan is fully repaid, so will be the depositors. This last assumption is introduced to reduce the number of types of clearing equilibria. Later, we will show that none of the main conclusions of the paper depend on any of the above assumptions. We only introduce them at this stage to reduce the number of cases that we need to consider.

2.1.3 Shocks

We assume that $\phi \in \mathbb{Z}$ banks are hit with primitive shocks. These shocks pertain to the profitability and repayment of loans banks make to debtors outside the interbank network — to households and firms for instance. We assume, without loss of generality, that a bank hit by a shock loses all of its revenues. Such a bank will have to sell its tangible assets in the secondary market. We then denote the probability that ϕ banks are hit by

[8]In Appendix E we examine the implications for our results of relaxing the main restrictions that we have imposed on the benchmark case considered in this and the next two sections of the paper.

these shocks as $p(\phi)$ for $\phi \in \{0, 1, ..., N\}$., where $\sum_{\phi=i}^{N} p(i) = 1$. This distribution is an object our analysis takes as given, which could be estimated by a policy or institutional decision maker using historical data.

2.1.4 Fire Sales

Denote the fire sale value of illiquid assets as $L \in [0, K]$. This value will depend on the price, which these assets are exchanged for in a secondary market. All else equal, one would expect a larger number of liquidating banks to translate into a lower equilibrium fire sale value. We can focus on the number of liquidating banks without loss of generality, since bank symmetry implies the total volume of liquidated assets is proportional this number. As such, we capture this relationship with a reduced-form liquidation function

$$L = \ell\left(\widehat{N}\right), \tag{1}$$

where $\widehat{N} \in \{0, 1, ..., N\}$ denotes the number of liquidations, $\ell(0) = K$, $\ell' < 0$ and $\ell'' > 0$ and $\ell(N) = 0$.

2.2 Equilibrium

To introduce the model's equilibrium in a transparent way, we commence the characterisation in the first subsection by assuming that $\phi = 1$ with certainty, (i.e. $p(\phi = 1) = 1$ and $p(\phi) = 0$ for $\phi \neq 1$). In the subsection subsequent, we relax this assumption by allowing for many shocks on the cycle.

2.2.1 Single Shock

The analysis here starts with the assumption that the cycle is sufficiently large, where the minimum size will be defined in what follows.

Proposition 1 *Let* $\delta = \frac{R-F+\ell(\widehat{N})}{F}$ *and let* v^* *denote the number of additional banks that are liquidated (that is other than the one initially hit by the shock). Let* $Int[x]$ *denote the integer part of a real number* x*. Then, for a sufficiently large cycle*

(a) if δ *is an integer* $v^* = \delta - 1$*, and*

(b) if δ *is not an integer* $v^* = Int[\delta]$*.*

Proof. We will first show that v^* satisfies the *Solvency Condition* (SC)

$$v^* R - (v^* + 1) F + v^* \ell(\widehat{N}) < 0 \leqslant (v^* + 1) R - (v^* + 2) F + (v^* + 1) \ell(\widehat{N} + 1). \tag{2}$$

where recall that $\ell(v^* + 1)$ takes the form in equation (1). We number the N banks of the cycle, so that the one hit by the shock is bank 0, any bank i ($i = 1, .., N - 1$) is a creditor to bank $i - 1$ and bank 0 is a creditor to bank $N - 1$. Suppose that the total number of banks that have already been liquidated, (including the one hit by the shock), is equal to λ and consider the status of bank $\lambda+1$. Then the partial repayment of bank λ to bank $\lambda+1$ is given by $D+(\lambda-1)R+\lambda\ell(\lambda)-\lambda F$. Given that depositors hold priority claims and since bank λ is liquidated, the repayment must be partial. The supposition that the cycle is large implies that then bank $N-1$ will fully repay its loan to bank 1. That is — since the cycle is large, the cascade ends before reaching bank $N-1$.

Given that all banks from bank 0 to bank λ are liquidated, the repayment to bank $\lambda+1$ must include all the assets of the liquidated banks. This includes the full repayment from bank $N-1$, all the revenues of the $\lambda-1$ banks not hit by the shock and the liquidation proceeds of all liquidated banks, minus the compensation received by all depositors with funds at the liquidated banks. Then, the total liquid assets of bank $\lambda+1$ are equal to $D+\lambda R-\lambda F+\lambda\ell(\lambda)$ and its liabilities are equal to $D+F$. Then, if $\lambda R-(\lambda+1)F+\lambda\ell(\lambda)<0$, bank $\lambda+1$ will be liquidated. We need to then consider two cases:

(a) if $\lambda R-(\lambda+1)F+(\lambda+1)\ell(\lambda+1)\geqslant 0$, bank $\lambda+1$ will fully repay bank $\lambda+2$, and therefore the number of banks liquidated is equal to $\lambda+1$. Given that $\lambda R-(\lambda+1)F+(\lambda+1)\ell(\lambda+1)<(\lambda+1)R-(\lambda+2)F+(\lambda+1)\ell(\lambda+1)$ and letting $v^*=\lambda$ we find that (2) holds.

(b) if $\lambda R-(\lambda+1)F+(\lambda+1)\ell(\lambda+1)<0$, bank $\lambda+1$ will not fully repay bank $\lambda+2$. Repeating the steps above we find that as long as $(\lambda+1)R-(\lambda+2)\,F+(\lambda+1)\,\ell(\lambda+1)=(v^*+1)R-(v^*+2)\,F+(v^*+1)\,\ell(v^*+1)\geqslant 0$, bank $\lambda+2$ will not be liquidated and thus (2) holds again.

The proof is completed by finding $\delta\in\mathbb{R}$ such that $\delta R-(\delta+1)F+\delta\ell(\delta)=0$. ■

A point to note here regarding the equilibrium is that its solution represents a fixed point problem. One can see this most easily from looking at the liquidation function in equation (1). The fire sale value of the illiquid assets depends upon the number of defaulting banks. But as discussed in the derivation of equation (2), the number of banks liquidating itself depends on the fire sale value of their assets.

Condition (2) holds as long as the cycle size is at least v^*+1. If the circle has either exactly v^*+1 banks and the last bank is unable to meet its obligations in full, or it has less than v^*+1 banks the settlements will depend on both v^* and the size of the circle. In this instance, the bank hit by a shock will also need to write-off some of the loan repayment coming from its debtor bank.

Suppose that $N\leqslant v^*+1$ and $(N-1)R-NF+N\ell(N)<0$. This second condition means that the last bank is liquidated and cannot meet its obligations. It also means that the value of total assets, excluding the inter-bank obligations, is not sufficiently high to cover the obligations to all depositors. The depositors of the banks that are not hit by a shock will be fully compensated. The depositors of the bank hit by the shock will be only partially compensated. The amount they receive will be equal to $(N-1)R-(N-1)F+N\ell(N)<F$. The next proposition summarises the above results:

Proposition 2 *Suppose that a bank belonging to a circle of size m is hit by a shock. Then* *the total number of banks in that circle that will be liquidated, $\hat{v}$, is given by*

$$\hat{v} = \begin{cases} v^* + 1 & \text{for} \quad N > v^* \\ N & \qquad\quad N \leqslant v^*. \end{cases} \tag{3}$$

Although aggregate losses will be higher when $N > v^*$, all depositors will be repaid in full. However when $N \leq v^*$, depositors of the bank hit by the shock will not be made whole.

This proposition formalises the notion of sufficiently large cycle size: when $N > v^*$. The second part of the proposition highlights a novel trade-off between aggregate losses and those to priority claimants. Although a smaller circle contains the size of a default cascade, the inability of the debtor bank to that hit by the shock to repay its obligation, diminishes the cashflow to the shocked bank's priority stakeholders. The above results for this simple model suggest that network structure and distribution of shocks are important determinants for the burden of losses across various types of claimholders from some primitive shock. We explore these issues in more detail in the sections that follow.

2.2.2 Multiple Shocks

Providing a general analysis of this case is a complicated combinatorial problem. For a given number of shocks, there exists a distribution across banks in the network. As such, we give a characterisation of the worst case scenario, where the space between each shocked bank is sufficiently large. Figure 2 gives two examples of shock realisations on a network with $N = 5$, $\phi = 2$ and $v^* = 1$. The left set of realisations in this example can be viewed as the worst case scenario. The two shocks are spaced-out, meaning the full cascade length of $v^* = 1$ is realised for each shock and total defaults are maximised at four. Given the implications for taxpayer funds, one can think of the upper-bound of this range as the appropriate object of analysis for particularly risk averse regulator or investor.

Let b_{i-s} and b_{i+t} denote any two banks hit by a shock. See then that the distance between these two banks is $t + s$. Notice that the solvency condition (2) still holds here in the context of multiple shocks as long as $v^* < s + t$. This will imply the liquidation function will assume the value $L = \ell(\phi[v^* + 1])$. That is — the total number of defaults equals the equilibrium cascade size v^* plus the shocked bank, repeated for each of the ϕ shocks. The next proposition summaries a consequence for aggregate losses, given that we consider the worst case scenario.

Lemma 3 *If $v^* \geqslant s + t$ for any two banks hit by shocks, then $\widehat{N} \leqslant \phi\,(v^* + 1)$.*

Proof. See Appendix B.1. ■

In light of Lemma 3, the range of possible scenarios for aggregate defaults lies somewhere over the range $[\phi + v^*, \phi(v^* + 1)]$. In the context of the right image in Figure 2, if two consecutive banks are hit with primitive shocks, the second bank will not necessarily receive their credits from the inter-bank market in full. In this instance, depositors of this second bank are less likely to receive their complete repayments, creating actionable issues for the deposit insurance agency. We now proceed to give a discussion to *ex ante* risk in the presence of an arbitrary distribution of shocks.

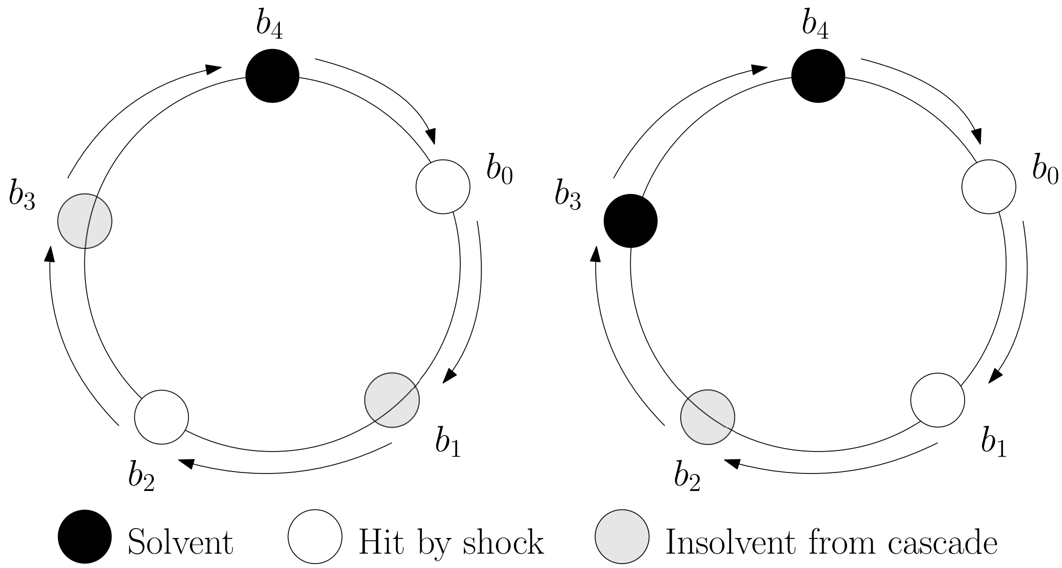

Figure 2: Directed cycle network where $N = 5$ and $\phi = 2$ and $v^* = 1$. Left figure shows scenario where shocks are spaced-out, right figure shocks scenario with consecutive shocks.

2.3 Risk

The structure of the model is such that, once one has found the equilibrium v^* as a fixed point, risk can be characterised in closed-form.

2.3.1 Survival Probability

Here we consider risk in terms of uncertainty regarding the number of primitive shocks to be realised, within the worst case scenario discussed in the previous subsection. In terms of balance sheet items, the uncertainty about revenues implies that all balance sheet items are risky, however, the level of risk varies significantly among them. In addition to the direct risk related to revenues, a bank faces two additional indirect risks on the assets side of its balance sheet. The first is related to the ability of its debtor bank to repay its loan — this depends on whether that bank is caught in the middle of a chain of bankruptcies — including both counterparty and network risk. The second risk is related to the value of its illiquid assets, which can drop below its book value due to fire sales. On the liability side, the commitments of the bank to its creditors are also risky. Depositors face lower risk given that they hold priority claims. Symmetry implies that the risk on the interbank loans on either side of the balance sheet bear the same risk. Equity holders, being residual claimholders, face the highest risk.

As a first step, we calculate the probability that a bank survives. Then we decompose the implicit risk between direct idiosyncratic risk due to a shock on revenues and indirect risk due to network effects. Recall that we assume a distribution $p(\phi)$. Let us now denote $v^*(\phi)$ as the number of of additional liquidations taking place in a cascade, when the number of banks hit by a shock are ϕ and the shocks are sufficiently far apart. We can then write the worst case probability of survival (denoted π) for any given bank as

$$\begin{aligned}\pi &= p(0) + p(1)\max\left(0, 1 - \frac{v^*(1)+1}{N}\right) + ... + p(N)\max\left(0, 1 - \frac{N\left(v^*(N)+1\right)}{N}\right) \\ &= p(0) + \sum\nolimits_{\phi=1}^{N} p(\phi)\max\left(0, 1 - \frac{\phi\left(v^*(\phi)+1\right)}{N}\right) \end{aligned} \tag{4}$$

where notice the max operator serves to ensure each term is weakly positive, as a probability. For instance, if there are N shocks, (one per bank), then unless $v^*(N) = 0$, the term $1 - N\left(v^*(N)+1\right)/N$ will be negative. Notice then that, since the liquidation function (1) is decreasing in the number of defaults, $v^*(\phi)$ will be an increasing function. Hence we can find a cutoff $\widehat{\phi} \in \mathbb{Z}$ such that $N/\widehat{\phi} = v^*(\widehat{\phi}) + 1$, meaning that for values of $\phi \geq \widehat{\phi}$, the probability of bank survival is zero. When combined with some further manipulations, we can re-write equation (4) as

$$\pi = \sum_{\phi=0}^{\text{Int}(\widehat{\phi})} p(\phi) - \sum_{\phi=1}^{\text{Int}(\widehat{\phi})} p(\phi)\frac{\phi}{N} - \sum_{\phi=1}^{\text{Int}(\widehat{\phi})} p(\phi)\frac{\phi}{N} - \sum_{\phi=1}^{\text{Int}(\widehat{\phi})} p(\phi)\frac{\phi}{N}\left[v^*(\phi) - 1\right] \tag{5}$$

where recall that the operator $\text{Int}(x)$ takes the largest integer directly below $x \in \mathbb{R}$. Notice that object ϕ/N represents the number of shocks per bank in the network; alternatively the probability of being hit by a shock. Equation (5) represents an analytical decomposition of the three types of risk banks face mentioned in the introduction. The first term on the right-side is the probability of the circle of being of size less than $\widehat{\phi}$ — this is a requirement for banks having any chance at survival under the worst case scenario. The second and third terms on the right-side give the idiosyncratic and counter-party risks. These two terms are identical — each represents the expected probability of being hit by a primitive shock — specifically, the risk a given bank or its debtor is hit. The final term gives network risk — the expected probability of being $v^*(\phi)$ links away from a bank hit by a shock and caught in the cascade of defaults that follows.

Expression (5) facilitates closed-form comparative statics on the network. Should one compare two networks with $v^*(\phi)$ being the same across all states ϕ, then it's clear that the network that places the biggest weight on high values of ϕ will have the lower survival probability. One should of course bear in mind that $v^*(\phi)$ is an equilibrium object depending on (2), which needs to be solved for recursively. One can consider two networks G and G' — both with the same number of banks N and the same distribution over shocks $p(\phi)$ — but different balance sheets. The latter point means that $v^*(\phi)$ will differ between G and G', as will $\widehat{\phi}$. Consequently, a comparison of survival probabilities depends on the probability mass below $\text{Int}(\widehat{\phi})$ for each network, as well as the equilibrium cascade size $v^*(\widehat{\phi})$.

2.3.2 Risk on Balance Sheet Items

Now having solved for and characterised the probability of survival in equation (5), we are in a position to evaluate the risk on each of the banks' claims. Firstly notice that a bank's revenues are guaranteed as long as they are not hit by a primitive shock. As such, we can define the risk premium on revenues as

$$\rho_R = \frac{1}{\pi_R}$$
$$\pi_R = 1 - \sum_{\phi=0}^{N} p(\phi)\frac{\phi}{N} \tag{6}$$

that is — the reciprocal of the probability of avoiding being hit by a shock. This is equal to the complement of the expected probability of getting hit by a primitive shock. When comparing two networks with the same number of banks, that which tilts the probability distribution towards a higher number of shocks yields a higher premium on revenues. Object π_R gives the probability that a bank does not get hit by a shock. Notice that this number must necessarily be above that of the bank's survival due to the presence of network effects. As such, we can then find the equilibrium probability of a bank failing due to network effects as the difference of (5) from (6)

$$\pi_R - \pi = \sum_{\mathrm{Int}(\widehat{\phi})+1}^{N} p(\phi) + \sum_{\phi=1}^{\mathrm{Int}(\widehat{\phi})} p(\phi)\frac{\phi}{N}\left[v^*(\phi)+1\right] - \sum_{\phi=0}^{N} p(\phi)\frac{\phi}{N}. \tag{7}$$

The first term on the right-side of equation (7) is the probability of being in a shock realisation where all banks in the cycle default. The second term is the probability of either being hit by a shock or caught in a default cascade in a state where there is a chance of survival. Then the third term is the expected probability of being hit by a shock. A network featuring a higher degree of potential financial distress, (e.g. where the liquidation function (1) decays at a faster rate), will generally have longer cascades, thereby increasing the first two terms in equation (7). We can then define the risk premium on interbank loans as

$$\rho_D = \frac{1}{1-(\pi_R - \pi)} \tag{8}$$

that is — the reciprocal of the probability of avoiding being caught in a default cascade. In deriving the premium for equity, we make a simplifying assumption that the last surviving bank's equity-holders are fully compensated. We consider this assumption relatively innocuous from a practical perspective, since the probability of being the last bank in a cascade is relatively low. Then we can define the equity premium as

$$\rho_E = \frac{1}{\pi} \tag{9}$$

that is — the reciprocal of the probability of survival. Comparing expressions (8) and (9), one can see that although network risk drives up both premia, $\rho_E > \rho_D$, as a consequence of priority structure. Our characterisation of the worst case scenario with regard to cascade chains means that depositors in this solution will always receive their full compensation. Within this simple model, we have derived closed-form expressions for *ex ante* risk faced by various stakeholders in banks. In the following two sections, we now apply this analysis to two network structures that capture important features of real financial networks.

3 Multiple Cycle Network

We now move to generalise the discussion of Section 2 to a network with many cycles of potentially differing sizes. As Jackson and Pernoud (2022) observe, the presence of directed cycles complicates substantially the analysis of networks. Without such cycles the network has a hierarchical structure and the clearing process is relatively straightforward. As we will see below the presence of directed cycles has also important implications for the distribution of losses following some initial shocks across a bank's claimholders. In this section, we abstract form issues related to hierarchies and focus on those pertaining to directed cycles. In the next section, we take a closer look on the impact of hierarchies on ex ante risk assessments

3.1 Environment

We retain most features of the single cycle network environment, with substantive departures detailed in what follows.

3.1.1 Architecture

We index the potential size of a given cycle as $m \in [2, N]$. The smallest possible cycle has two banks, which lend to each other, while the largest contains all banks in the network. We define $n(m)$ as the number of cycles with m banks and N^* as the total number of cycles. Two accounting identities then follow

$$N = \sum_{m=2}^{N} mn(m)$$
$$N^* = \sum_{m=2}^{N} n(m),$$

where for N even $N^* \in [1, \frac{N}{2}]$ and for N odd $N^* \in [1, \frac{N-1}{2}]$.[9] Despite the above restrictions the number of potential networks of size N is very large and can be calculated recursively.[10] Figure 3 gives a graphical example of a simple network with three cycles.

3.1.2 Shocks

Given the degree of complexity in this framework with multiple cycles, we start the analysis with the following simplifying assumptions.

[9] For N odd the maximum number of possible cycles is equal to $\frac{N-1}{2}$ with $\frac{N-2}{2}$ cycles of size 2 and 1 cycle of size 3.

[10] See Appendix A for the derivation.

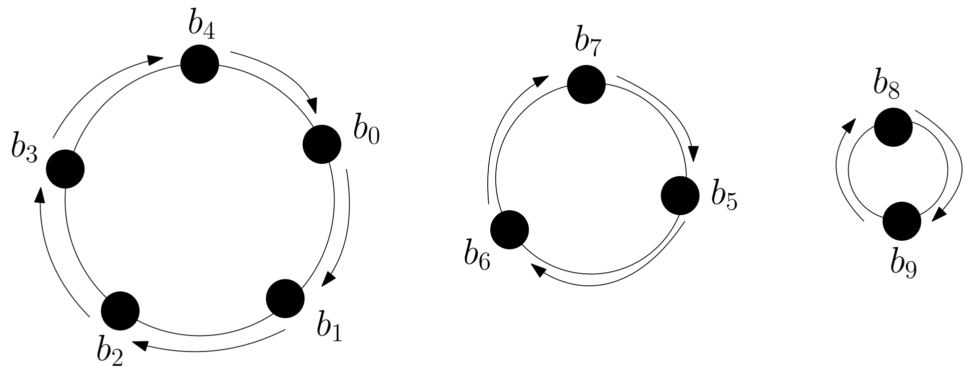

Figure 3: Network with $N = 10$, $N^* = 3$, $n(2) = 1$, $n(3) = 1$, $n(5) = 1$.

Assumption 3: There is no aggregate uncertainty about initial shocks.[11]

Therefore, we assume that exactly $\phi < N^*$ banks will fail and, thus, the *ex ante* probability that a bank will fail is equal to ϕ/N. However, as we will demonstrate below even if there is no aggregate uncertainty with respect to initial shocks there is still aggregate uncertainty with respect to final outcomes..

Assumption 4: At most one bank in each circle can be hit by a shock. The probability that a circle is hit by a shock is independent of its size.

As long as N^* is much larger relatively to ϕ, this assumption has minimal impact. As discussed in Section 2, when multiple shocks are allowed to hit a given circle, the degree of amplification can increase. As long as the worst case scenario holds for a multi-shock circle, (where the full default cascade is realised), our results of this section remain unchanged.

3.1.3 Fire Sales

We maintain the assumption that the liquidation values of assets are given as in expression (1). However we emphasise that this value is not cycle-specific. That is — the same liquidation value for assets applies to banks in financial distress in any cycle.

3.2 Equilibrium

The results of Proposition 1 and Proposition 2 continue to hold in the model presented in this section. In computing the equilibrium, the fixed point problem must account for liquidations across the entire network, rather than in a particular cycle. The effect of

[11] As we did in Section 2, it is straightforward (although computationally more demanding) to extend the analysis to the case when there is aggregate uncertainty. Under aggregate uncertainty, the number of shocks ϕ is a discrete random variable that can take any integer value in the interval $[0, m]$ where m is circle size. By repeating the analysis below for all possible values, multiplying the results for each value with its corresponding relative frequency and summing up we can obtain the corresponding expected values.

the ϕ shocks, on the the total number of insolvencies depends on (a) the structure of the network, that is its partition into circles, and (b) the distribution of shocks across these circles. Let $\hat{n}(m)$ denote the number of circles of size m that were hit by a shock and let $\hat{N}$ denote the total number of banks in the economy that will be liquidated. Notice that $\sum_{m=2}^{N} \hat{n}(m) = \phi$. Then, the total number of banks in the economy that will be liquidated is given by

$$\hat{N} = \sum_{m=2}^{v^*} m\hat{n}(m) + (v^* + 1) \sum_{m=v^*+1}^{N} \hat{n}(m) \equiv \Psi(v^*). \tag{10}$$

Notice that v^* is non-increasing in liquidation values. As liquidation values drop the number of banks being liquidated either stays the same or goes up. In the second case the right-hand side of (10) increases as all banks of even bigger circles get liquidated. Since our model features endogenous fire sales, the right-side of (10) is also a function of $\hat{N}$, denoted as $\psi\left(\hat{N}\right)$, given that v^* depends on the liquidation value of assets. Thus, the number of banks that will be liquidated in equilibrium, $\hat{N}^*$, is given by the fixed points of

$$\hat{N} = \psi\left(\hat{N}\right) = \Psi\left(V\left(\ell\left(\hat{N}\right)\right)\right) \tag{11}$$

where the last expression recognises that changes in $\hat{N}$ will directly affect liquidation values through (1), which in turn will affect v^* through $V(L)$, a step-function that can be derived using the solvency condition (2), which in turn will affect $\psi\left(\hat{N}\right)$ through $\Psi(v^*)$. We defer more details regarding this structure's solution to Appendix C. We also consider extensions to this network structure in Appendix E.

3.3 Risk

Here we assume that the network structure is known, but a particular bank's position within the network is not known (i.e. the size of its corresponding circle). For an individual bank, they will likely have detailed information regarding their counter-parties, but it is unlikely they can see information about banks further up the chain. In Appendix E.4, we also consider the simpler case where the bank knows the size of its cycle.

3.3.1 Survival Probability

In what follows, we characterise the probability of survival for cascade lengths no more than 3 banks ($v^* \leq 3$). The solution can easily be extended to chains of greater length, but the number of cases that must be counted significantly rises.

As mentioned previously, in sufficiently large cycles hit by a shock, the last surviving bank will only be partially compensated. Similarly to the single cycle model, we will approximate the solution where the last surviving bank is fully compensated. In Appendix E, we extend the model method to allow for such partial repayments and show that they indeed have minimal impact on our results.

In this version of the model with multiple cycles, there are several states of the world, which are indistinguishable in terms of final outcomes. Here we leverage this point to simplify the construction of the solution. What matters is how many of the ϕ shocks hit cycles of a given size, not which particular cycles were hit. We let μ denote the number of distinct circle sizes. In what follows, we index states of the world, (in terms of the cycles hit by shocks), by s.

Lower liquidation values materialise when shocks hit larger circles, leading to more bank failures and higher values of v_s^*. As such, given that circles have at least two banks, if $v_s^* = 0$ or $v_s^* = 1$ for some s, then $v_s^* = 0$ or $v_s^* = 1$, respectively in every s. This follows since the highest possible number of liquidations is either 1 (for $v_s^* = 0$) or 2 (for $v_s^* = 1$) for every circle hit by a shock, bringing the total number of liquidations to either ϕ or 2ϕ, respectively. That is, the total number of liquidations does not depend on the distribution of shocks. In contrast, at higher values, v_s^* can vary from state to state, since the number of banks liquidated and therefore the size of fire sales, will depend on the distribution of shocks.[12]

Now we break-down the approximate solution for $v_s^* \leq 3$ into two parts. First we consider the set of scenarios whereby $v_s^* = v^* \ \forall s$, meaning that the default cascade length does not very by state. As stated above, in the cases of $v^* \in \{0, 1\}$, this holds by design. This may not always be true for $v_s^* \in \{2, 3\}$, but in cases where parameters imply that it is, we can write the survival probability as

$$\pi = \left(1 - \frac{\phi}{N^*}\right) + \frac{\phi}{N^*}\left(\sum_{m=v^*+2}^{N}\left(\frac{mn(m)}{N}\frac{m - [v^* + 1]}{m}\right)\right) \tag{12}$$

for $v^* \in \{0, 1, 2, 3\}$. The first term in expression (12) is the probability a cycle is not hit by a primitive shock. The second term states that, conditional on being hit by a shock, one must consider all the possible cycle sizes, ranging from $v^* + 2$ through N. This follows since smaller cycle sizes see all banks eliminated in the event of a shock, (see Proposition 2). Inside the summation in (12), one considers the probability of being inside a cycle of a given size m, $mn(m)/N$ and the probability of surviving conditional on being in such a cycle, $[m - (v^* + 1)]/m$.

We now consider scenarios whereby v_s^* is not a constant for all possible states s. Given our focus on cascades less than 3 in length, the case we need to consider is parameterisations that lead to states with $v_s^* \in \{2, 3\}$, varying with s. We denote the total number of composite states for the network by $\bar{s}$. Given that the liquidation function (1) is decreasing in the total number of liquidations, there exists a state say γ, such that for all $s < \gamma$, $v_s^* = 3$ and for all $s \geqslant \gamma, v_s^* = 2$ (if one orders states in descending order of total liquidations). The probability of survival can then be written as

$$\pi = \left(1 - \frac{\phi}{N^*}\right) + \frac{\phi}{N^*}\left(\sum_{s=0}^{\gamma} p_s \sum_{m=5}^{N}\left(\frac{mn(m)}{N}\frac{m-4}{m}\right) + \sum_{s=\gamma+1}^{\bar{s}} p_s \sum_{m=4}^{N}\left(\frac{mn(m)}{N}\frac{m-3}{m}\right)\right) \tag{13}$$

[12] Appendix D gives some examples of how one would go about enumerating the set of states to be considered.

where $p_s = S^I/\binom{N^*}{\phi}$, that is equal to the number of indistinguishable states divided by the total number of states. The first double summation in the second bracket of equation (13) captures the likelihood of surviving given that the circle is hit by a shock and $v_s^* = 3$, while the second is for $v_s^* = 2$. When $v_s^* = 3$ all banks in circles of size less than or equal to 4 are liquidated. Thus, to survive a bank must belong to a circle of size greater than or equal to 5 and at a distance of at least 4 from the bank hit by the shock. When $v_s^* = 2$ all banks in circles of size less of equal to 3 are liquidated. To survive, a bank must belong to a circle of size greater or equal to 4 and at a distance of at least 3 from the bank hit by the shock. Taken together, (12) and (13) summarise the approximate solution for the probability of survival when $v_s^* \leq 3$.

The parsimonious structure of our model and the simple expressions for survival probabilities, facilitate closed-form comparative statics on network architecture. The following proposition gives an expression showing how the approximate survival probabilities vary as the distribution of cycles changes.

Proposition 4 *(Comparative statics on architecture). Consider two networks G and G', which have the same number of banks N and shocks ϕ, but different architectures (configurations of cycles). Specifically, G and G' have different distributions of cycles $\{n(m)\}_{m=2}^N$ and $\{n'(m)\}_{m=2}^N$, respectively. Assume that the equilibrium cascade length is the same $1 \leq v^* \leq 3$, for all possible states across G and G'. The difference in survival probabilities $\Delta \equiv \pi(G') - \pi(G)$ is given by*

$$\Delta = \frac{\phi}{N} \sum_{m=2}^{v^*+1} (m - [v^* + 1]) \left(\frac{n(m)}{N^*} - \frac{n'(m)}{N^{*\prime}} \right) \tag{14}$$

Proof. See Appendix B.2. ■

The expression on the right-side of (14) is the difference in network risk across G' and G. Notice that if $v^* = 0$, then the object $\Delta = 0$, given that banks are only faced with idiosyncratic risk. Recall the term ϕ/N gives the number of shocks per bank. The sum spans the region of cycle sizes, whereby all banks in a circle hit will default. Notice that any cycle hit in either network with size v^*+2 or above will give the same outcome in terms of number of defaults and hence the contribution to the survival probability. The last term in parentheses gives the difference in the fraction of cycles of size m across networks G and G'. When there are more cycles of smaller size in network G, cascade lengths are constrained relative to G', leading to a higher likelihood of any given bank surviving, from an *ex-ante* viewpoint. We now turn to study the impact of network risk on balance sheet items.

3.3.2 Risk on Balance Sheet Items and Deposit Insurance

In the context of this model with multiple cycles, notice that we can write the risk premium on revenues as in (6), but with probability

$$\pi_R = 1 - \frac{\phi}{N}.$$

Similarly to section 2.3.2, we can then find the probability of bank failure due to network effects, in the case where $v_s = v^* \leq 3\,\forall s$ as

$$\pi_R - \pi = \frac{\phi}{N^*} \sum_{m=v^*+2}^{N} \frac{mn(m)}{N} \left(1 - \frac{1}{m} - \frac{v^*}{m}\right),$$

where we've invoked expression (12). See that the subtracted term $1/m$ inside the summation reflects counter-party risk, while v^*/m reflects network risk. Similarly, when considering cascades varying by state $v_s^* \in \{2, 3\}$, we can write this network effect using (13) as

$$\pi_R - \pi = \frac{\phi}{N^*} \left(\sum_{s=0}^{\gamma} p_s \sum_{m=5}^{N} \left(\frac{mn(m)}{N} \left[1 - \frac{1}{m} - \frac{3}{m}\right]\right) + \sum_{s=\gamma+1}^{\bar{s}} p_s \sum_{m=4}^{N} \left(\frac{mn(m)}{N} \left[1 - \frac{1}{m} - \frac{2}{m}\right]\right)\right)$$

where the terms carry similar meaning to the previous expression. One can then use these expressions for $\pi_R - \pi$ with (8) in order to find the risk premium on inter-bank liabilities. Similarly we can use expressions (12) and (13) with (9) to get the premium on equity.

Recall in Section 2, we assumed that with multiple shocks, they were spaced-out sufficiently to allow full cascade chains to be realised. This marks the substantial point of departure when considering single shocks on many possible cycles. In the current context, we can make statements about risk to the depositors, since potential exists for those of shocked banks to lose funds in the event of a full cycle revolution. Recall, this happens specifically when $m \leq v^* + 1$, (Proposition 2). We can derive an upper bound for the risk premium of deposits that corresponds to the case where the depositors of banks hit by shocks do not receive any compensation. This assumption simplifies the manipulations, while also giving a conservative estimate, which is an important object from a regulatory perspective. We can write this expression for the upper-bound on the premium as

$$\rho_F \leq \frac{1}{\pi_F} \tag{15}$$

where the expression for π_F depends on how v_s^* varies across states s. When the cascade length is the same for all states $v_s^* = v^* \in \{1, 2, 3\}$, we can write this as

$$\pi_F = \left(1 - \frac{\phi}{N^*}\right) + \frac{\phi}{N^*} \left\{ \sum_{m=2}^{v^*+1} \frac{mn(m)}{N} \frac{m-1}{m} + \sum_{m=v^*+2}^{N} \frac{mn(m)}{N} \right\}. \tag{16}$$

That is — depositors receive full funds when their cycle is not hit by a shock — the first term in (16). The first summation says, conditional on their cycle being hit, they receive full funds if they are not the bank hit by the shock in a small cycle ($2 \leq m \leq v^* + 1$), which happens with probability $(m-1)/m$. The second summation says they receive their full stake also if they're cycle is shocked but is of a sufficiently large size ($m \geq v^* + 2$). In the case where $v_s^* \in \{2, 3\}$ for a given network, we can write

$$\pi_F = \left(1 - \frac{\phi}{N^*}\right) + \frac{\phi}{N^*} \left(\begin{array}{c} \sum_{s=0}^{\gamma} p_s \left(\sum_{m=5}^{N} \frac{mn(m)}{N} + \sum_{m=2}^{4} \frac{mn(m)}{N} \frac{m-1}{m} \right) + \\ \sum_{s=\gamma+1}^{\bar{s}} p_s \left(\sum_{m=4}^{N} \frac{mn(m)}{N} + \sum_{m=2}^{3} \frac{mn(m)}{N} \frac{m-1}{m} \right) \end{array} \right). \tag{17}$$

Looking firstly at the double summations inside the parentheses of (17). The first set of double summations considers when $v_s^* = 3$ $(s \leq \gamma)$ — the depositors receive full funds if in a circle bigger than size 5 or if they're not the shocked bank in sizes 2, 3 or 4. The second set follows the same logic for $v_s^* = 2$ $(s > \gamma)$ states.

Taken together, (15), (16) and (17) characterise the *ex-ante* risk premium associated with deposits in the network. It is almost immediate to see how this probability varies across different networks. For instance, in (16), one can see that the distribution of banks across small cycles relative to large is crucial, summarised by $\sum_{m=v^*+2}^{N} \frac{mn(m)}{N}$. A skew of the distribution towards smaller cycles $2 \leq m \leq v^* + 1$ shifts weight towards scenarios whereby depositors lose funds due to full cycle revolutions of cascades.

The above analysis highlights the sensitivity of *ex ante* risk assessments to the structure of the financial network. In particular, the structure of the network affects the distribution of expected losses across different types of claimholders, classified by their priority status. This has important implications (a) for the design of deposit insurance premia, and (b) for the cascade of bank failures across the system.

3.3.3 Equity Premium Puzzle

Around four decades ago, Mehra and Prescott (1985) observed that the risk premium on U.S. equities exceeds by an order of magnitude what should be expected for the premium to be from neo-classical finance. The article planted the seeds for an extensive literature offering various solutions to the puzzle.[13] In our model, risk premia respond to two sources of risk. One is common idiosyncratic risk, capturing any risk arising within an entity (in our case revenues from loans granted to housolds and banks). The second type of risk is due to network effects arising because of potential losses resulting from the liquidation of other entities belonging to the same network. While all liability holders are exposed to this second type of risk, those who hold priority claims are less exposed.

In our model, the gross equity risk premium in the absence of cascades is given by the premium due to idiosyncratic shocks, ρ^R. The corresponding risk premium on priority claims is equal to 1 (in the absence of cascades these claims are fully repaid). The corresponding overall risk premia that also include the risk due to cascades are given by ρ^E and ρ^F. The latter, especially if we allow for partial repayments, is very close to 1. In contrast, ρ^E is significantly higher than ρ^R. Thus, the difference between equity and priority claims, the risk premium, increases when we account for a premium for systemic risk. We now illustrate these effects through a numerical example.

Example 1 *Consider a network with* 100 *banks and structure described in Table 2. The balance sheet parameters are:* $R = 1$, $F = 0.95$, $K = 0.25$, $D = 0.8$, *and* $\phi = 3$. *The liquidation function is given by* $\ell\left(\hat{N}\right) = K\left(1 - \sqrt{\frac{\hat{N}}{N}}\right)$. *The results are shown in Figure 3 where we compare the number of liquidations for alternative network structures. Thus, each line shows for each network structure the distribution of liquidations across all states*

[13] For a useful review the reader is referred to Mehra (2006).

of nature. The total number of states for this example are equal to $\binom{N^*}{\phi} = \binom{27}{3} = 2,925$. *For these parameter values* $v^* = 4$.

Size	E1	E1A	E1B	E1C
2	6	0	0	0
3	8	12	0	0
4	7	7	16	0
5	3	3	3	3
6	1	1	1	1
7	1	1	1	1
8	1	1	1	9
π_R	0.970	0.970	0.970	0.970
π	0.896	0.887	0.872	0.850
π_F	0.973	0.974	0.974	0.994
ρ_R	1.031	1.031	1.031	1.031
ρ_D	1.080	1.090	1.109	1.136
ρ_E	1.117	1.127	1.147	1.176
ρ_F	1.027	1.027	1.026	1.006

Table 2: Network for Example (E) 1 and variant examples. The network configuration changes progressively with Example 1A (E1A), 1B (E1B) and 1C (E1C). Numbers in the top panel pertain to the distributions of cycles. Numbers in the bottom panel give the calculated survival probabilities and risk premia the configurations, shocks and balance sheets imply.

Here we run an experiment where we start with Example 1's network structure and then progressively re-allocate the banks from the smallest cycles to larger ones. Initially we break-up the cycles of size 2 and re-distribute the displaced banks into cycles of size 3 — reflected in Example 1A. We then do the same with the circles of 3s in Example 1B, then finally with 4s in Example 1C. Note that the 4 circles can be re-distributed in anyway in moving to Example 1C, since all states are indistinguishable due to the $v^* = 4$ cascade length.

Figure 4 depicts the distributions of aggregate losses for the examples. As we change the network structure, distributions that first-order dominate have a higher number of average liquidations. For the network of Example 1, we find that the lower number of liquidations, 6, is obtained when all shocks hit cycle of size 2 and the higher number of liquidations, 15, is obtained when all shocks hit cycle of sizes greater or equal to 5. When all shocks hit cycles of size 2, the total number of liquidations cannot exceed 6. As we move to states of nature where shocks hit larger cycles, liquidations increase. Then as we move to Example 1A, we we find that the lower number of liquidations is equal to 9 while the higher number is still 15. The key observation is that as the distribution of cycle sizes moves to the right so does the distribution of liquidations across the states of nature. The distribution continues to shift rightwards as we move from Example 1A to Example 1B. Finally for Example 1C, again since $v^* = 4$, it does not matter which cycles are hit by shocks. Independent of cycle size, there are 5 liquidations in each one of those cycles. For the same reason, further shifts of the distribution of cycle sizes will not affect the number of liquidations.

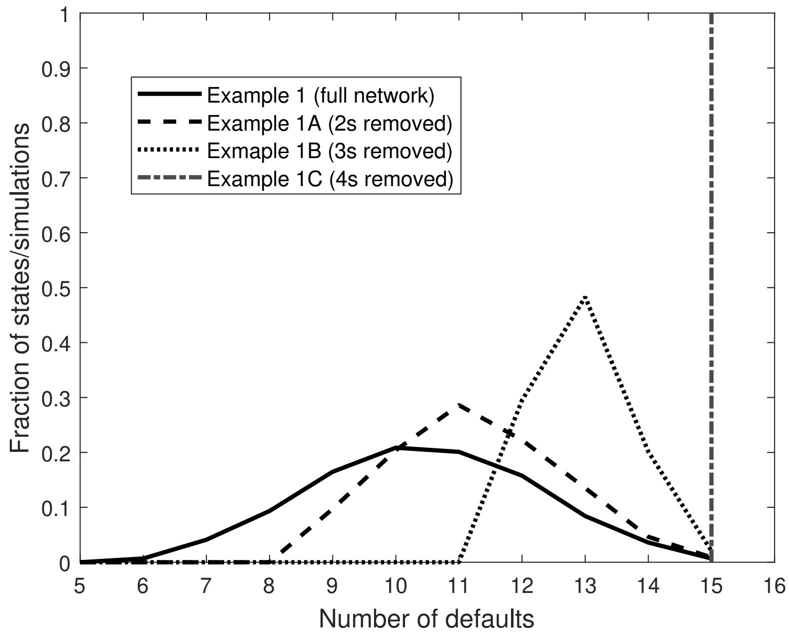

Figure 4: Distribution of aggregate defaults for Example 1 and re-configurations.

Table 2 also presents probabilities of survival as well as implied risk premia. Notice the non-linearities in these numbers as the removed circles increase in size. The probability of survival π decreases at an increasing rate, falling by 0.9% as the 2s are removed, 1.5% as the 3s are removed and 2.2% as the 4s are removed. This translates into an equity risk premium that increases at an increasing rate — rising by 1%, 2% and nearly 3% when removing the 2s, 3s and 4s respectively. Notice also that the conservative premium on deposits decreases as one shifts the distribution of losses to the right, with the premium in Example 1C being slightly above unity due to approximation error. After having focused on cycle networks until now, we now move to study the implications for *ex-ante* risk of hierarchies by analysing a network structure known as 'complete order'.

4 Complete Order Network

In the last section, we showed that the presence of directed cycles in the financial network has significant implications for the distribution of expected losses across various bank claimholders. In order to keep the analysis tractable, we have abstracted from issues arising as a result of hierarchies in the network. There are some banks that occupy central locations in the network by providing funds to many other banks, while there are also banks with many creditors. Under such hierarchical systems, it is clear that the risks that

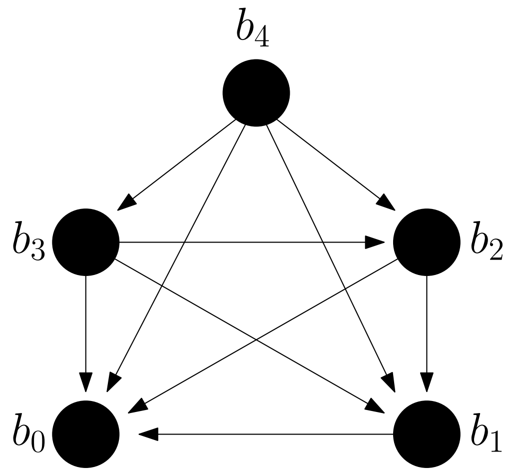

Figure 5: Complete order network with $N = 5$.

depositors face depend on the position of their bank in the network. In order to obtain a better understanding for the implications that such hierarchies have, not only for the stability of the financial system, but also for regulators who are interested in protecting depositors, we turn now our focus to a network that is strictly hierarchical.

4.1 Environment

Here we outline the points of departure from the model of Section 2.

4.1.1 Architecture

In this section, we adopt stricter notation in labelling banks. We index the number of creditors one institution has with integer $c \in \{0, 1, ..., N-1\}$; a bank is identified with b_c. There complete asymmetry — we assume that for each possible value of c, there is associated exactly one bank. Figure 5 displays an example structure with $N = 5$. The bank at the top b_4 has four creditors and zero debtors, while bank b_0 has four debtors and zero creditors.

4.1.2 Balance Sheets

Assets	**Liabilities**
Revenues (R_c)	Deposits (F)
Bank Loans ($[N-c-1]D$)	Bank Deposits (cD)
Non-liquid Assets (K)	Equity (E)

Table 3: Balance sheet for bank b_c.

The balance sheet of bank b_c is presented in Table 3. Notice that the value of revenues R_c is asymmetric across banks. Bank 0 has interbank loans as assets in the amount of $(N-1)D$ and liabilities of 0. In contrast, bank b_{N-1} has interbank assets of 0 and liabilities of $(N-1)D$. All other banks are on a scale in between these two extremes. We assume multiplicity of the factor D in order to simplify the algebra. One should also notice that, due to common elements F, K, E across banks, it must also follow that

$$R_c - R_{c-1} = 2D \tag{18}$$

when comparing the balance sheets of banks c and $c-1$. That is — revenues must be increasing as one moves up the chain in considering banks with higher amounts of interbank liabilities.

4.1.3 Shocks

Similarly to Section 2, to keep the analysis tractable in terms of cases, we assume a single shock, that can hit any bank in the network. Each bank is assumed an equal probability $1/N$ of being the recipient. In the event of being hit by a shock, bank b_c's revenues are reduced to zero. One other point to emphasise: given the asymmetry of balance sheets, it is possible for a bank hit by a shock here to stay solvent. This contrasts with the networks of Sections 2 and 3, where cancellation of the D interbank assets and liabilities means, with certainty, that the bank hit with the shock will default.

4.2 Equilibrium

The following proposition exploits the ordering structure of the network, to make transitive statements regarding liquidations and solvency.

Proposition 5 *Suppose that $R_j \to 0$ and b_j is liquidated. Then if $R_k \to 0$ and $k > j$, b_k will be liquidated. Conversely, if $R_m \to 0$ and b_k for some $k < m$ is not liquidated then bank b_{k-1} will also not be liquidated.*

Proof. See Appendix B.3. ■

Proposition 5 allows us to limit the number of banks that we need to directly consider, when it comes to characterising equilibrium cascade lengths. The following proposition summarises the equilibrium solvency condition, which goes in place of (2) for this particular network structure.

Proposition 6 *The equilibrium cascade length when bank c is hit by a shock, v_c^*, is characterised by solvency condition*

$$R_{c-v_c^*} + (N - 2c + 2[v_c^* - 1])D + \sum_{s=0}^{v_c^*+1} D_{c-s}(\ell(v_c^* + 1)) - F < 0 \leq$$
$$R_{c-v_c^*-1} + (N - 2c + 2[v_c^*])D + \sum_{s=0}^{v_c^*+2} D_{c-s}(\ell(v_c^* + 1)) - F$$

where

$$D_{c-s}(\ell(v_c^*+1)) = \frac{\max\left(0, \sum_{j=1}^{v_c^*-1} R_{c-j} + (N-c-1)D + \sum_{j=0}^{v_c^*-1} D_{c-j}(\ell(v_c^*+1)) + \ell(v_c^*+1) - F\right)}{c-s}$$

Proof. See Appendix B.4. ■

There are several points to notice in Proposition 6. First, see that the equilibrium cascade length in this model becomes bank-specific. The object D_{c-s} is the repayment of bank $c-s$ to each of its individual creditors. The payment D_{c-s} is an accumulation of revenues of previous defaulters, excepting that hit by the shock, its own interbank assets, previous partial payments from defaulters, its liquidation value less its obligation to depositors. The following proposition shows that cascades of defaults are increasing in length as one moves up the chain towards banks that have more creditors.

Proposition 7 *Inequality* $v_k^* \geqslant v_j^*$ *holds* $\forall k > j$.

Proof. See Appendix B.5. ■

4.3 Risk

We assume that banks know the structure of the entire network. As such, given their own interbank assets and liabilities, they can exactly identify their position within the network.

4.3.1 Survival Probability

Since the survival probability is position-specific, we denote it by π^c for a bank with c creditors

$$\pi^c = \frac{1}{N}\mathbb{1}_{[(N-c-1)D \geq F+cD]} + \sum_{s=0}^{c-1}\frac{1}{N} + \sum_{s=c+1}^{N}\frac{1}{N}\mathbb{1}_{[v_s^* < (s-c)]} \tag{19}$$

where note that $\mathbb{1}_{[x]}$ represents an indicator, which equals 1 when argument x is true and 0 otherwise. The first term on the right-side of expression (19) relates to bank c's idiosyncratic risk. Should the interbank repayments it receives be insufficient to cover its liabilities, in the absence of revenues, it will fail. The second term captures the probability of the shock hitting a bank with fewer creditors than c. When a bank further down the chain is hit, it has no bearing on banks at the top of the chain since debts run downwards, not upwards. The final term in (19) relates to network risk. There is a $1/N$ chance of each bank s above c in the ordering being hit by the shock; bank c will feel the effect of that if the equilibrium cascade size is at least as long as the distance from c.

Proposition 8 *(Comparative statics on position). Consider two distinct banks* c *and* c'*. Assume without loss of generality that* $c' > c$*. The difference in survival probabilities*

$\Delta' = \pi^c - \pi^{c'}$ *is given by*

$$\Delta' = \frac{1}{N}I + \left\{ \sum_{s=c+1}^{c'} \frac{1}{N}\mathbb{1}_{[v_s^* < (s-c)]} - \sum_{s=c}^{c'-1}\frac{1}{N} + \sum_{s=c'+1}^{N}\frac{1}{N}\left\{\mathbb{1}_{[v_s^* < (s-c)]} - \mathbb{1}_{[v_s^* < (s-c')]}\right\} \right\} \tag{20}$$

where $I = \mathbb{1}_{[(N-c-1)D \geq F+cD]} - \mathbb{1}_{[(N-c'-1)D \geq F+c'D]}$.

Proof. See Appendix B.6. ■

The expression for Δ' in (20) is comprised of two sets of terms. The first set is made-up of the first term I/N, which is the difference in idiosyncratic risk across c and c'. Object I is an integer defined as the difference in the indicators for solvency of c and c' in the event of being hit by the primitive shock. As a consequence of Proposition 5, it must be the case that I is a binary variable, assuming value 0 if both banks survive or both fail in the event of receiving the shock, or value 1 if c survives while c' fails. The second set of terms, inside the curly parentheses, represents the difference in network risk across the two positions. Network risk is broken-down into three components. The first component, (second term in (20), gives the chance of survival for bank c is any bank between itself and bank c' is hit by the shock. The second term is the safety afforded to bank c' by being higher up the chain of obligations. The final term gives the incremental survival probability for c when any bank above c' is hit by a shock. See that by Lemma 7, it must be that this last term is weakly positive. We next consider changes in architecture.

Proposition 9 *(Comparative statics on architecture). Consider two networks G and G', with $N+1$ banks and N banks, respectively. All other details are identical. The difference in survival probabilities for a given position c, denoted as $\widehat{\Delta} = \pi^c(G) - \pi^c(G')$ is given as*

$$\widehat{\Delta} = \frac{1}{N}I' + \sum_{s=0}^{c-1}\left[\frac{1}{N+1} - \frac{1}{N}\right] + \sum_{s=c+1}^{N}\left[\frac{1}{N+1} - \frac{1}{N}\right]\mathbb{1}_{[v_s^* < (s-c)]} + \frac{1}{N+1}\mathbb{1}_{[v_{N+1}^* < N+1-c]} \tag{21}$$

where

$$I' = \frac{N}{N+1}\mathbb{1}_{[(N-c)D \geq F+cD]} - \mathbb{1}_{[(N-c-1)D \geq F+cD]}.$$

Proof. See Appendix B.7. ■

Notice that adding more banks to the network for a fixed position, as in Proposition 9, amounts to adding more banks at the top of the network; more banks with greater interbank obligations than bank c. The first term on the right-side of (21) represents the chance of being hit by a shock and failing. Note that in network G', bank c has one extra debtor, meaning default in the event of a shock becomes weakly less likely. However, the chance of being hit by the shock becomes lower. The second term gives the difference in the probability of a shock below position c in the chain. The third is the difference in survival probability when a shock hits above position c and the last covers the extra contingency for survival for the additional bank added at the top of the network (with $N+1$ creditors).

Lastly, one can also see how changes in balance sheets, holding position constant, affect survival probabilities immediately by looking at the last term in (19). For instance, consider an increase in the parameter F. The difference in survival probability between two networks, G and G' (higher F) would be $\sum_{s=c+1}^{N} \frac{1}{N}\{\mathbb{1}_{[v_s^*<(s-c)]} - \mathbb{1}_{[v_s^{*\prime}<(s-c)]}\}$, where the first term pertains to G and the second to G'. The effect of such a change is highly non-linear and highlights the role of bank heterogeneity in this network structure. The change in F may affect the equilibrium cascade length of some banks but not others; the difference $\mathbb{1}_{[v_s^*<(s-c)]} - \mathbb{1}_{[v_s^{*\prime}<(s-c)]}$ may be zero or negative and varies with s.

4.3.2 Risk on Balance Sheet Items and Deposit Insurance

Risk on revenues can simply be written as $\rho_R = 1/\pi_R$ where $\pi_R = (N-1)/N$ in this context. We can then find the probability of failing due to network effects as

$$\pi_R - \pi^c = \frac{N-1-c-\mathbb{1}_{[(N-c-1)D \geq F+cD]}}{N} - \sum_{s=c+1}^{N} \frac{1}{N}\mathbb{1}_{[v_s^*<(s-c)]}. \tag{22}$$

The first term on the right-side of equation (22) shows the probability of network failure falls in c, since banks hit below are not along the chain of c's creditors. The probability is also falling in the second term — as cascades starting at s get shorter, the chance of reaching c drops. We again use this expression (22) in (8) in deriving the risk on interbank liabilities, as well as the expression (19) in (9) for the risk on equity.

We can price deposit insurance in terms of the expected losses to the corresponding stakeholders. The depositors of a bank $s \geq 0$ positions below that hit by a shock receive cash flows

$$F_{c-s} = \min\left(F, R_{c-s}\mathbb{1}_{c-s\neq 0} + (N-c-1)D + \sum_{j=0}^{s-1} D_{c-j}(\ell(v_c^*+1)) + \ell(v_c^*+1)\right). \tag{23}$$

That is — they receive F if the bank is solvent or has sufficient cashflows post-liquidation. Otherwise they receive their bank's revenues, payments in full from solvent debtor banks, partial payments from insolvent debtor banks and the firesale value of the bank's illiquid assets. Using equation (23), in addition to the setup for shocks, we can write the expected losses for a depositor of the bank in position c as

$$\widehat{F}_c = F - \frac{1}{N}F_c - \frac{cF}{N} - \frac{1}{N}\sum_{s=c+1}^{N}\left\{\mathbb{1}_{v_s^*<(s-c)}F + \mathbb{1}_{v_s^*\geq(s-c)}F_{c-s}\right\},$$

which is simply the difference between the face value and the expected repayment. The second term in $\widehat{F}_c$ is the repayment if bank c is hit, the third term shows the full face value is repaid when banks below bank c are hit and the final term relates to the contingencies of banks above c being hit.

What is clear from this expression for $\widehat{F}_c$ is that the expected losses to depositors are highly contingent on position. Moreover the effect of moving along the hierarchy has an ambiguous effect on $\widehat{F}_c$. Although bank $N-1$ faces no network risk, it stands to lose the most revenues if hit by the primitive shock, which catalyses the largest cascade effect (Proposition 7). In a relatively small network (low N) with a rapidly decreasing liquidation function (1), it is possible that bank $N-1$ could face higher expected losses than those with fewer creditors. As Acemoglu *et al.* (2015) observe having more creditors introduces a diversification effect as losses are spread out to many counterparties. However, if initial losses are very large this also raises the possibility of a higher number of bank failures.[14]

Our analysis in this section suggests that the risks faced by depositors depend on the position of their bank in the interbank network. In our network, all banks have the same number of links but there are stark differences between them in terms of the nature of these links. Focusing at the the two banks at the extreme ends of the network we make the following observations. Depositors at bank $N-1$ are not exposed to risks associated with the performance of other banks. However, their bank's performance could affect all the others, as well as their depositors. In contrast, depositors at bank 0 can potentially be affected by the failure of any bank in the system. This raises some questions about deposit insurance risk premia. A bank's decision to participate in the interbank network affects the riskiness of its liabilities, not only through risks directly related to its counterparties, but also those related to the structure of the whole network. As Zawadowski (2013) observes, when banks hedge their portfolios, they do not take into account the negative externalities that their actions exert on the rest of the banking network. Here, we make an analogous argument in relation to deposit insurance.[15]

5 Concluding Comments

Contagion risk poses a serious threat to worldwide financial stability, as shown in the Global Financial Crisis of 2007/08 and more recently with the 2023 United States/European banking crisis. These types of events place significant strain on public funds and must be priced accurately by regulators in advance. This paper contributes to the literature through being to first to formalise the relationship between *ex ante* assessments and network risk.

We presented two different network structures, which capture features of real networks, yet are sufficiently simple to retain tractability. Our model facilitates closed-form risk premia expressions, as well as yielding analytical comparative statics. Our results show that risk is a highly non-linear object, which which can not be accurately estimated when abstracting from network effects. A counter-intuitive headline result is that lower aggregate losses can come in tandem with higher losses for priority claimholders.

Given that our framework is stylised, it has some limitations. Firstly, our model uses a

[14] For a more extensive discussion of diversification in financial networks, see Battiston *et al.* (2012).

[15] In this work we have concentrated on risks due to default. However, there is an alternative network risk due to lending freezes resulting from a loss of confidence in the market. In that case bank 0 could potentially be the one affecting all other banks, through reluctance to renew its lending contracts. See Acemoglu *et al.* (2021) for a network analysis of bank freezes.

financial equilibrium approach, where we have taken balance sheets and hence the network structure as fixed. While this methodology can be useful for a regulator who would like to know the underlying risks of a networked system, it might not be entirely appropriate for the design of policies. As Beale *et al.* (2011) stress, regulators also need to take into consideration how a new policy might alter the incentives of participants to form links.[16] A future potential application could be to understanding risk across international networks with multinational banks.[17] Financial system turbulence is an issue that affects all entities in a developed economy — from workers to entrepreneurs and to CEOs of global corporations. It is hoped that this paper's first step towards understanding the relationship between network and balance sheet risk will spur much more such research activity to come.

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[16] For endogenous formation see Acemoglu and Azar (2020) for production networks and Babus (2016) for financial networks.

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Appendix

A Network Space

Note that this is a modified version of the procedure described in `https://janmr.com/blog/2008/12/twelve-ways-of-counting/` We begin by calculating the number of possible networks for a given number of circles, N^*. This problem is equivalent to finding the number of ways of allocating N unlabeled balls into N^* unlabeled urns. The allocation problem is equivalent to the number of ways of writing the integer N as the sum of N^* positive integers each integer greater or equal to 2. The problem reduces to finding the number of ways of allocating $N - N^*$ balls into N^* urns and then add 1 ball in each urn. The arrangements we get before we add the extra balls are called partitions of $N - N^*$ into N^* parts. Let

$$\left| \begin{matrix} N - N^* \\ N^* \end{matrix} \right|$$

denote the number of partitions. We have

$$\left| \begin{matrix} \alpha \\ \alpha \end{matrix} \right| = \left| \begin{matrix} \alpha \\ 1 \end{matrix} \right| = 1, \alpha \geqslant 1 \text{ and } \left| \begin{matrix} \beta \\ \alpha \end{matrix} \right| = 0, \alpha > \beta > 0,$$

and the boundary conditions

$$\left| \begin{array}{c} 0 \\ 0 \end{array} \right| = 1 \text{ and } \left| \begin{array}{c} \alpha \\ 0 \end{array} \right| = 0, \alpha > 1.$$

For the general case of partitioning α into β parts, we split the partitions into those that have at least one 1 among the parts and those where each part is greater than 1. The first group of partitions is obtained by including all partitions of $n-1$ into $m-1$ parts, and then add 1 in the empty arrangement, and the second group of partitions is obtained by including all partitions of $n-m$ into m parts, where 1 could be added to each part. We then have

$$\left| \begin{array}{c} N-N^* \\ N^* \end{array} \right| = \left| \begin{array}{c} N-N^*-1 \\ N^*-1 \end{array} \right| + \left| \begin{array}{c} N-2N^* \\ N^* \end{array} \right|.$$

By repeating this procedure we will end up with a sum of 0s and 1s.

The above calculation derives the number of possible networks when the number of circles are restricted to be equal to N^*. Then, the number of all possible networks is given by

$$\sum_{N^*=1}^{\frac{N}{2}} \left| \begin{array}{c} N-N^* \\ N^* \end{array} \right| \text{ for } N \text{ even and } \sum_{N^*=1}^{\frac{N-1}{2}} \left| \begin{array}{c} N-N^* \\ N^* \end{array} \right| \text{ for } N \text{ odd.}$$

B Proofs

B.1 Lemma 3

Proof. We show this for the case when $\phi = 2$ and the two banks hit by a shock are neighbors, say b_i and b_{i+1}.[18] Let $\hat{v}$ be equal to the number of additional liquidations had the distance between the the two banks been sufficiently large. In that case the total number of liquidations would have neen given by $\widehat{N} = 2(\hat{v}+1)$ and the solvency condition given by

$$\hat{v}R - (\hat{v}+1)F + \hat{v}\ell(2(\hat{v}+1)) < 0 \leqslant (\hat{v}+1)R - (\hat{v}+2)F + (\hat{v}+1)\ell(2(\hat{v}+1)). \quad (24)$$

Bank b_i will repay $D + \ell(\widehat{N}) - F$ to bank b_{i+1}. Clearly if b_i has been liquidated then b_{i+1} will also be liquidated given that its loan to b_i has only been partially repaid. Following the same steps as those used for the derivation of (2), we find that the number of additional liquidations following the liquidation of bank b_{i+1}, $\bar{v}$, is given by the modified solvency condition

$$\bar{v}R - (\bar{v}+2)F + (\bar{v}+1)\bar{v}\ell(\bar{v}+2) < 0 \leqslant (\bar{v}+1)R - (\bar{v}+3)F + (\bar{v}+2)l(\bar{v}+2). \quad (25)$$

The total number of liquidations are now given by $\widehat{N} = \bar{v}+2$. Comparing (24) with (25), we find that on both sides there is an extra F term and an extra $\ell(\widehat{N})$ term. However, the inequality $R - F - \ell(\widehat{N}) > 0$ implies that $\hat{v} \leqslant \bar{v} \leqslant \bar{v}+1$. Thus, $\bar{v}+2 \leqslant 2(\hat{v}+1)$. ■

[18] It will become clear that if the result holds for this case it must also hold for any other arrangement of banks on the cycle.

B.2 Proposition 4

Proof. Recall

$$\pi = \left(1 - \frac{\phi}{N^*}\right) + \frac{\phi}{N^*}\left(\sum_{m=v^*+2}^{N}\left(\frac{mn(m)}{N}\frac{m-[v^*+1]}{m}\right)\right).$$

So define the difference

$$\begin{aligned}\Delta &\equiv \pi' - \pi \\ &= \phi\left(\frac{1}{N^*} - \frac{1}{N^{*\prime}}\right) + \frac{\phi}{N}\sum_{m=v^*+2}^{N}(m-[v^*+1])\left(\frac{n(m)'}{N^{*\prime}} - \frac{n(m)}{N^*}\right). \qquad (26)\end{aligned}$$

Then see that

$$\begin{aligned}\frac{N}{N^*} &= \frac{\sum_{m=2}^{v^*+1}(m-[v^*+1])n(m)}{N^*} + \frac{\sum_{m=v^*+2}^{N}(m-[v^*+1])n(m)}{N^*} \\ \Rightarrow \frac{\sum_{m=v^*+2}^{N}(m-[v^*+1])n(m)}{N^*} &= \frac{N}{N^*} - \frac{\sum_{m=2}^{v^*+1}(m-[v^*+1])n(m)}{N^*}. \qquad (27)\end{aligned}$$

Moreover

$$\begin{aligned}\sum_{m=2}^{N}\frac{n(m)}{N^*} &= 1 \\ \Rightarrow \sum_{m=v^*+2}^{N}\frac{n(m)}{N^*} &= 1 - \sum_{m=2}^{v^*+1}\frac{n(m)}{N^*}. \qquad (28)\end{aligned}$$

Hence we can use (27) and (28) in (26) to get

$$\begin{aligned}\Delta &= \phi\left(\frac{1}{N^*} - \frac{1}{N^{*\prime}}\right) + \frac{\phi}{N}\sum_{m=v^*+2}^{N}(m-[v^*+1])\left(\frac{n(m)'}{N^{*\prime}} - \frac{n(m)}{N^*}\right) \\ &= \frac{\phi}{N}\left\{\left(\frac{N}{N^*} - \frac{N}{N^{*\prime}}\right) + \sum_{m=v^*+2}^{N}(m-[v^*+1])\left(\frac{n(m)'}{N^{*\prime}} - \frac{n(m)}{N^*}\right)\right\} \\ &= \frac{\phi}{N}\sum_{m=2}^{v^*+1}(m-[v^*+1])\left(\frac{n(m)}{N^*} - \frac{n'(m)}{N^{*\prime}}\right)\end{aligned}$$

■

B.3 Proposition 5

Proof. If $R_c = 0$, b_c will be liquidated if $(N-c-1)D < F + cD$ or $(N-2c-1)D < F$. If $R_{c-1} = 0$, b_{c-1} will be liquidated if $(N-c)D < F + (c-1)D$ or $(N-2c+1)D < F$. The proof follows by comparing the two insolvency conditions and by induction. The second part of the lemma follows from (18) and by comparing (29) and (30) below. ■

B.4 Proposition 6

Proof. Consider bank b_c, where $R_c = 0$. Denote aggregate losses by object Λ.

If $(N-c-1)D \geqslant F + cD$ or

$$(N-2c-1)D \geqslant F \tag{29}$$

then $\widehat{N} = 0$ & $\Lambda = R_c$ & $v_c^* = 0$

If $(N-2c-1)D < F$ then

If $(N-2c-1)D + \ell(1) \geqslant F$ then $\hat{N} = 1$

& $\Lambda = R_c + K - \ell(1)$ & $v_c^* = 0$

If $(N-2c-1)D + \ell(1) < F$ then[19]

$F \to \min\{(N-c-1)D + \ell(v_c^*+1), F\}$ and

$D_c \to \max\{0, (N-c-1)D + \ell(v_c^*+1) - F\}/c$

Consider bank b_{c-1}.

If $R_{c-1} + (N-c-1)D + D_c \geqslant F + (c-1)D$ or

$$R_{c-1} + (N-2c)D + D_c \geqslant F \tag{30}$$

then $\widehat{N} = 1$ & $\Lambda = R_c + K - \ell(1)$ & $v_c^* = 0$

If $R_{c-1} + (N-2c)D + D_c < F$ then

If $R_{c-1} + (N-2c)D + D_c + \ell(2) \geqslant F$ then[20]

$\widehat{N} = 2$ & $\Lambda = R_c + 2\left(K - \ell(2)\right)$ & $v_c^* = 1$

If $R_{c-1} + (N-2c)D + D_c + \ell(2) < F$ then

$F \to \min\{R_{c-1} + (N-c-1)D + D_c + \ell(v_c^*+1), F\}$ and

$D_{c-1} \to \max\{0, R_{c-1} + (N-c-1)D + D_c + \ell(v_c^*+1) - F\}/(c-1)$

Consider bank b_{c-2}

If $R_{c-2} + (N-c)D + D_c + D_{c-1} \geqslant F + (c-2)D$ or

$$R_{c-2} + (N-2c+2)D + D_c + D_{c-1} \geqslant F \tag{31}$$

then $\hat{N} = 2$ & $\Lambda = R_c + 2\left(K - \ell(2)\right)$ & $v_c^* = 1$.

If $R_{c-2} + (N-c+2)D + D_c + D_{c-1} < F$ then

If $R_{c-2} + (N-c+2)D + D_c + D_{c-1} + \ell(3) \geqslant F$ then

$\hat{N} = 3$ & $\Lambda = R_c + 3\left(K - \ell(3)\right)$ & $v_c^* = 2$

If $R_{c-2} + (N-c+2)D + D_c + D_{c-1} + \ell(3) < F$ then

$F \to \min\{R_{c-2} + (N-c)D + D_c + D_{c-1} + \ell(v_c^*+1), F\}$ and

$D_{c-2} \to \max\{0, R_{c-2} + (N-c)D + D_c + D_{c-1} + \ell(v_c^*+1)\}/(c-2)$.

The solvency condition then follows by induction. ■

[19] At this point is not known the total number of banks that will be liquidated, $\hat{N}$, given that v_c^* has not been determined yet.

[20] Lemma 5 ensures that the liquidation value in that case is equal to $\ell(2)$.

B.5 Proposition 7

Proof. This follows from (a) as c increases R_c increases and therefore the impact of catastrophic shocks increase, and (b) the number of borrowing banks, and thus repayments, declines. ■

B.6 Proposition 8

Proof. The definition of the difference is

$$\Delta' = \frac{1}{N}\mathbb{1}_{[(N-c-1)D \geq F+cD]} + \sum_{s=0}^{c-1} \frac{1}{N} + \sum_{s=c+1}^{N} \frac{1}{N}\mathbb{1}_{[v_s^* < (s-c)]}$$
$$- \frac{1}{N}\mathbb{1}_{[(N-c'-1)D \geq F+c'D]} - \sum_{s=0}^{c'-1} \frac{1}{N} - \sum_{s=c'+1}^{N} \frac{1}{N}\mathbb{1}_{[v_s^* < (s-c')]}.$$

The proof then just follows easily using the assumption that $c' > c$. ■

B.7 Proposition 9

Proof. The definition of the difference is

$$\Delta' = \frac{1}{N+1}\mathbb{1}_{[(N-c)D \geq F+cD]} + \sum_{s=0}^{c-1} \frac{1}{N+1} + \sum_{s=c+1}^{N+1} \frac{1}{N+1}\mathbb{1}_{[v_s^* < (s-c)]}$$
$$- \frac{1}{N}\mathbb{1}_{[(N-c'-1)D \geq F+c'D]} - \sum_{s=0}^{c'-1} \frac{1}{N} - \sum_{s=c'+1}^{N} \frac{1}{N}\mathbb{1}_{[v_s^* < (s-c')]}.$$

The expression then follows from further basic manipulations. ■

v^*	$\Psi(v^*)$
0	ϕ
1	2ϕ
2	$2\hat{n}(2) + 3(\phi - \hat{n}(2))$
3	$2\hat{n}(2) + 3\hat{n}(3) + 4(\phi - \hat{n}(2) - \hat{n}(3))$
$\vdots$	$\vdots$
v^*	$2\hat{n}(2) + \cdots + v^*\hat{n}(v^*) + (v^*+1)(\phi - \hat{n}(2) - \cdots - \hat{n}(v^*))$

Table 4: The function $\Psi(v^*)$

C Details on Multiple Cycles Network Equilibrium Solution

Here we provide more details on solving for the model's equilibrium.

The Function $\Psi(v^*)$ We begin the analysis of (11) by taking a closer look at the function $\Psi(v^*)$ given by (10). Table 4 shows the values of this function for different values of v^*. When $v^* = 0$, the only banks that get liquidated are those that were hit by a shock. Given that the minimum size of a circle is 2 and given that each circle can be hit by at most one shock, when $v^* = 1$ the number of banks that will be liquidated will be twice the number of initial shocks. For $v^* \geqslant 2$, as long as there are circles of size less than v^* hit by a shock, then the total number of liquidations on such circles is restricted by their size. For example, for $v^* = 3$ all the banks of circles of size less or equal to 3 that were hit by a shock will be liquidated, as shown by the first two terms in the table. In contrast, for all circles of size greater or equal than 4 that were hit by a shock exactly 4 banks will be liquidated, as shown by the last term of the table.

By collecting terms we get[21]

$$\Psi(v^*) = (v^* + 1)\phi - \sum_{\gamma=1}^{v^*-1} (v^* - \gamma)\,\hat{n}(\gamma + 1). \tag{32}$$

Clearly, (32) implies that both the total number of banks that will get liquidated and the size of aggregate losses will depend on the structure of the network.

The Function $V(L)$ Next, we turn our attention to the function $V(L)$. Given that at least ϕ banks will be liquidated, the maximum value of liquidated assets is equal to $\ell(\phi) < K$. Then, from the discussion of account settlements, we find that if $R - 2F + \ell(\phi) \geqslant 0$ then $v^* = 0$. Let δ be a real number such that the LHS of the solvency condition (2) is equal to 0, that is $\delta R - (\delta + 1) F + \delta L = 0$. Then,

$$v^* = V(L) = Int(\delta) = Int\left[\frac{F}{R - F + L}\right] \tag{33}$$

[21] We can write the bottom expression of the table as $(v^* + 1)\phi - (v^* - 1)\hat{n}(2) - (v^* - 2)\hat{n}(3) - \cdots - \hat{n}(v^*)$.

The Solution By substituting (1) in (33) and (33) in (32) we obtain a closed form expression for (11). Depending on the parameter values of the model, we can get two types of solutions shown in Figure 6 and described below.

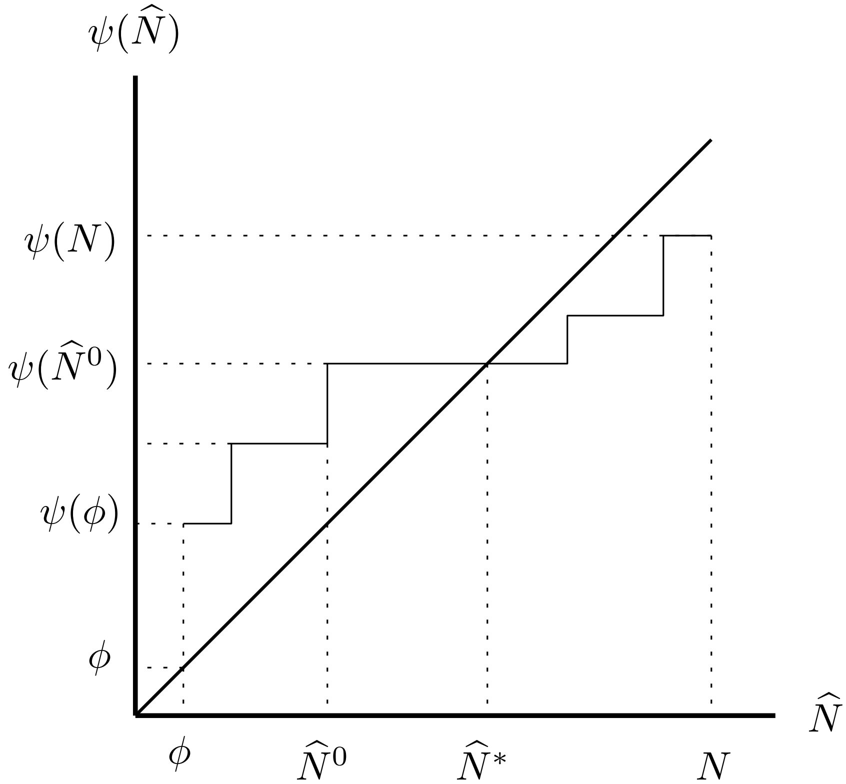

Figure 6: The function $\psi(\hat{N})$

Proposition 10 *Given a network G and ϕ shocks the number of banks that are liquidated in equilibrium are given by:*

(a) If $R - 2F + \ell(\phi) \geqslant 0$ then $\hat{N}^* = \phi$; (Type 1: Without-Cascades Equilibrium)

(b) If $R - 2F + \ell(\phi) < 0$ then there exists $\hat{N}^* > \phi$ such that (11) is satisfied; (Type 2: With-Cascades Equilibrium)

In part (a), the inequality implies that if the only banks liquidated are those hit by the initial shock, the liquidation value of the assets is sufficiently high for $v^* = 0$ (no cascades) which in turn means that $\hat{N}^* = \phi$. Figure 6 shows the determination of the equilibrium where the 45^0 line stands for the left side of (11) and the bold step function stands for the right side of (11). When $R - 2F + \ell(\phi) < 0$ then for $\hat{N} = \phi$ we have $v^* > 0$. As the number of liquidated banks increases, the liquidation value of assets drops. However, given that v^* can only take integer values, as long as the second inequality in the solvency condition (2) are satisfied, $V\left(\ell\left(\hat{N}\right)\right)$ remains stays the same (we move horizontally). Given that $\phi < n^*$ and therefore $\hat{N}^* < N$, the step function will eventually cross the 45^0 line. The crossing (there might be multiple) corresponds to the equilibrium

$s^I = \hat{n}(2)$	S^I	$\hat{N}_s$
0	$1 \times \binom{n^*-n(2)}{\phi}$	3ϕ
1	$n(2) \times \binom{n^*-n(2)}{\phi-1}$	$2+3(\phi-1)$
2	$\binom{n(2)}{2} \times \binom{n^*-n(2)}{\phi-2}$	$4+3(\phi-2)$
$\vdots$	$\vdots$	$\vdots$
ϕ	$\binom{n(2)}{\phi} \times 1$	2ϕ

Table 5: States for $v_s^* = 2$.

of the model. From Figure 6 we find that when $\hat{N} = \hat{N}^0$, $\psi^0 = \psi\left(\hat{N}^0\right) > \hat{N}^0$. However, $V\left(\ell\left(\psi\left(\hat{N}^0\right)\right)\right) = V\left(\ell\left(\hat{N}^*\right)\right)$ which implies that $\psi\left(\hat{N}^0\right) = \hat{N}^*$.

Remark 1 *When the parameters are such that case (b) of Proposition 10 is relevant there might be multiple crossings of the step function with the 45^0 line. In that case, the equilibria can be ranked according to their corresponding liquidation values.*[22] *When there are multiple equilibria our solution above always identifies the 'best' one, that is the one with the smaller number of liquidations (higher liquidation values).*

D State Calculations for Multiple Cycles Network

In this appendix, we briefly illustrate how one goes about enumerating all possible states of the world, which enter into risk calculations. When considering a network with balance sheets yielding $v_s^* = 2$ for all states s, we can list the possible states as in Table 5. In the context of Subsection 3.3.2, $\hat{n}(2)$ is the number of circles of size 2 that are hit by a shock and s^I denotes an arbitrary such state. Object S^I is the number of indistinguishable states for a given s^I and $\hat{N}_s$ is the number of banks that default in a given state. Table 6 gives performs the same enumeration for a network where states can vary between $v_s^* \in \{2, 3\}$. We consider a specific example below.

Example 2 *Consider a network with $N = 21$, $n^* = 7$ and $\mu = 4$, where $n(2) = 3$, $n(3) = 2$, $n(4) = 1$ and $n(5) = 1$. Further, $\phi = 3$. For this example $S = \binom{n^*}{\phi} = 35$.*

For the network of Example 2 for $\hat{n}(2) = (0, 1, 2, 3)$ the number of corresponding states are equal to $(4, 18, 12, 1)$ which add up to 35. For $\hat{n}(2) = 2$, there are 4 banks that will be liquidated, since they belong to the two circles of size 2 hit by shocks. For each of the remaining circles hit by shocks there will be 3 banks liquidated. The corresponding list of states is given in Table 7.

[22] For a more thorough discussion of multiplicity see Jackson and Pernoud (2022).

$s^I = (\hat{n}(2), \hat{n}(3))$	S^I	$\hat{N}_s$
$0 = (0, 0)$	$1 \times 1 \times \binom{n^*-n(2)}{\phi}$	4ϕ
$1 = (0, 1)$	$1 \times n(3) \times \binom{n^*-n(2)-n(3)}{\phi-1}$	$3 + 4(\phi - 1)$
$2 = (1, 0)$	$n(2) \times 1 \times \binom{n^*-n(2)-n(3)}{\phi-1}$	$2 + 4(\phi - 1)$
$3 = (0, 2)$	$1 \times \binom{n(3)}{2} \times \binom{n^*-n(2)-n(3)}{\phi-2}$	$6 + 4(\phi - 2)$
$4 = (1, 1)$	$n(2) \times n(3) \times \binom{n^*-n(2)-n(3)}{\phi-2}$	$5 + 4(\phi - 2)$
$5 = (2, 0)$	$\binom{n(2)}{2} \times 1 \times \binom{n^*-n(2)-n(3)}{\phi-2}$	$4 + 4(\phi - 2)$
$6 = (1, 2)$	$\binom{n(2)}{1} \times 1 \times \binom{n^*-n(2)-n(3)}{\phi-2}$	$8 + 4(\phi - 3)$
$\vdots$	$\vdots$	$\vdots$
$s' = (\hat{n}(2), \hat{n}(3))$	$\binom{n(2)}{\hat{n}(2)} \times \binom{n(3)}{\hat{n}(3)} \times \binom{n^*-n(2)-n(3)}{\phi-\hat{n}(2)-\hat{n}(3)}$	$2\hat{n}(2)+3\hat{n}(3)+4(\phi-\hat{n}(2)-\hat{n}(3))$
$\vdots$	$\vdots$	$\vdots$
$\bar{s} = (\phi, 0)$	$\binom{n(2)}{\phi} \times 1 \times 1$	2ϕ

Table 6: States for $v_s^* \leqslant 3$.

$s = (\hat{n}(2), \hat{n}(3))$	S_s^I	$\hat{N}_s$
$0 = (0, 1)$	2	11
$1 = (1, 0)$	3	10
$2 = (0, 2)$	2	10
$3 = (1, 1)$	12	9
$4 = (2, 0)$	6	8
$5 = (1, 2)$	3	8
$6 = (2, 1)$	6	7
$7 = (3, 0)$	1	6

Table 7: States for $v_s^* \leqslant 3$ for Example 2.

E Extensions for Multiple Cycles Network

E.1 $D < F$

Now we allow for the obligation to the supplier to be less than the obligation to the depositors. When the bank that is hit by a shock cannot meet fully its obligations, its creditor's compensation will be equal to $\max\{0, D + L - F\}$. We will first show that if $D + L - F > 0$ then the analysis above is still valid. Then, we will show the changes that we need to make when $D + L - F < 0$.

Suppose that $D + L - F > 0$, meaning that the creditor bank receives a payment. Clearly, if the creditor bank can fully meet its obligations to its depositors and its creditor, the analysis remains the same. If it cannot, then its creditor will be given $R+D-2F+2L > 0$, where the inequality follows from $D+L-F > 0$ and $R > F$. It follows that all depositors in subsequent rounds will also be fully compensated and thus the analysis is still the same.[23]

Next, suppose that $D + L - F < 0$. Now the creditor of the bank hit by the shock receives nothing. Then the creditor's assets assets are equal to $R+L$. As long as $R \geqslant D+F$, it will fully meet its obligations. If $R < D + F$ then the bank will be liquidated but it will still fully meet its obligations as long as $R+L \geqslant D+F$. If, in contrast, $R+L-D-F < 0$ its own creditor bank will receive $R+L-F$. The assets of this creditor bank are equal to $2R + 2L - F$. Then as long as $2R + L - F \geqslant D + F$, it will fully meet its obligations. If $2R + L - F < D + F$, then it will be liquidated but will still fully meet its obligations as long as $2R+2L-F \geqslant D+F$. If, in contrast, $2R+2L-D-2F < 0$, its own creditor bank will receive $2R+2L-2F$. By induction we conclude that the number of additional banks that are liquidated, v^*, (that is other than the one initially hit by the shock) satisfies the new *Solvency Condition* (SC)

$$v^*R - v^*F - D + v^*L < 0 \leqslant (v^* + 1)\,R - (v^* + 1)\,F - D + (v^* + 1)\,L. \tag{1'}$$

Once more, this condition holds as long as the size of the circle is at least $v^* + 1$.

E.2 Multiple Shocks

One of the aims of this paper is to demonstrate how, for a given network structure, the distribution of shocks across the network is an important determinant of the number of liquidations in equilibrium and thus the level of aggregate losses. In the benchmark case, we have assumed that each circle can be hit by at most one shock. It was briefly mentioned that, while this assumption simplified a lot of the account settlements analysis, it has also weakened our results. In this section, we allow for multiple shocks to hit a given circle and show that this can strengthen the impact of the distribution of shocks on aggregate outcomes.

The introduction of multiple shocks enlarges the state space. Now the total number of states is equal to $\binom{N}{\phi} = \frac{N!}{\phi!(N-\phi)!}$. We are going to focus on the case where $D < F$. The

[23] For circles with size smaller than $v^* + 1$ the analysis also remains the same given that all creditor banks do not receive any compensation.

implication of this restriction is that the creditor of a bank hit by a shock, will not receive any compensation, even if the latter bank's debtor survives. In this case, the assets of the bank hit by the shock are equal to D, which are not sufficient to cover the claims of depositors.[24]

Next consider what happens for a fixed value of v^* when a circle is hit by a second shock. For circles of size less than $v^* + 1$, all banks get liquidated when even only one of them is hit by a shock. A second shock will not affect the number of liquidated banks, but it will have distributional effects. In contrast, when a circle of size greater than $v^* + 1$ is hit by a shock then the number of banks that are liquidated will depend on both the exact size of the circle and the position of the two banks on the circle. The assumption $D < F$ implies that the number of banks liquidated after a shock hits, is independent of what happened to the shocked bank's creditor (i.e. whether the creditor was hit by a shock or was partially or fully compensated). Clearly, the number of banks that will be liquidated will be between $v^* + 2$ (this will be the case when the two banks hit by shocks are next to each other) and $2(v^* + 1)$ (when the shortest path between them is at least $v^* + 1$.

The above discussion implies that, when we introduce multiple shocks, the distribution of the number of liquidated banks across states is affected in two ways. Firstly its variance increases since there are more ways to have shocks concentrated on small size circles, as there are similarly for large circles. Secondly, this variance effect is asymmetric, as the upper bound of liquidations has now increased. Furthermore, so far we have assumed that the value of v^* has not been affected with this extension. However, as the losses due to fire sales increase with the number of liquidations, v^* can be even higher in those states where there are multiple shocks in large circles. In summary, allowing for multiple shocks exacerbates the impact of network effects on systemic risk and aggregate outcomes.

E.3 Risk Premium for D under Partial Compensation

In the benchmark case, we have assumed that all banks that are not liquidated have the interbank loans that they offered to other banks fully repaid. However, in circles where $m > z^* + 1$, the first bank to survive ($z^* + 1$ places down from the bank hit by the shock) will only be partially compensated. The partial compensation will be state dependent and we denote it as $D^s < D$. We will consider the more complicated case where $v_s^* \in \{2, 3\}$. Let $E(D^P)$ denote the expected value of compensation to be received from the debtor bank when we allow for partial compensation. Then, we have

[24]For the case when $D > F$, the analysis is more complicated. However it will become clear that, in that case, our conclusions about the impact of the shock distribution on aggregate outcomes would be even stronger.

$$E(D^P) = \left(\left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*} \left(\begin{array}{c} \sum_{s=0}^{\gamma} p_s \sum_{m=6}^{N} \left(\frac{mn(m)}{N} \frac{m-5}{m} \right) + \\ \sum_{s=\gamma+1}^{\bar{s}} p_s \sum_{m=5}^{N} \left(\frac{mn(m)}{N} \frac{m-4}{m} \right) \end{array} \right) \right) D$$

$$+ \quad \frac{\phi}{n^*} \left(\sum_{s=0}^{\gamma} p_s \sum_{m=5}^{N} \frac{mn(m)}{N} \frac{1}{m} D_s + \sum_{s=\gamma+1}^{\bar{s}} p_s \sum_{m=4}^{N} \frac{mn(m)}{N} \frac{1}{m} D_s \right)$$

The first term on the right-hand side is equal to the probability that the bank will survive conditional on receiving full compensation times the value of the claim. The term $1 - \frac{\phi}{n^*}$ is equal to the probability of survival conditional that the circle where the bank belongs is not hit by a shock. The next term in brackets is equal to the probability of survival conditional on (a) that the circle where the bank belongs is hit by a shock, and (b) the bank is fully compensated. In (13), the second condition was absent given that all banks were assumed to be fully compensated. The first double summation is over those states where $v^* = 3$. Then, banks whose circles where hit by a shock and were fully compensated by their buyers must belong to circles of size gretaer than 6.[25] With probability $\frac{mn(m)}{N}$, the bank belongs to a circle of size m and given that 4 banks are liquidated and one survives but is only partially compensated, the probability that the bank survives and is fully compensated is equal to $\frac{m-5}{m}$. The second double summation is over those states where $v^* = 2$ and is derived in a similar way.

The second term on the right hand side is equal to the expected value of partial compensation. The expressions are similarly derived but there are two differences. In circles of size equal to $v^* + 2$ that were hit by a shock there is one surviving bank that is partially compensated. For this reason the counter has decreased by 1. The second is that in circles of size greater than $v^* + 2$ there is only one bank that will receive partial compensation and this accounts for the term $\frac{1}{m}$.

Under partial compensation, the implicit risk premium on inter-bank liabilities, ϱ^{D^P}, is given by

$$\varrho^{D^P} = \frac{D}{E(D^P)}.$$

E.4 The Bank's Position in the Network is Known

In the main text, we have assumed that agents know the structure of the network but do not know the exact position (circle size) of their bank in the network. Let m^* denote the size of the circle that a given bank belongs to. If $m^* \leqslant v_s^* + 1$ then the bank will only survive if its circle is not hit by a shock, which happens with probability $1 - \frac{\phi}{n^*}$. If

[25] In circles of size 5 only one firm survives but is only partially compensated.

$m^* > v_s^* + 1$ the probability that the bank survives is now given by:[26]

$$\pi_{v_s^*} = \left(1 - \frac{\phi}{n^*}\right) + \frac{\phi}{n^*} \frac{m^* - (v_s^* + 1)}{m^*}.$$

In the case when the bank's position was unknown we had to take the expectation of the last term over all possible circle sizes. Once more we can use the probabilities of survival to calculate the corresponding risk premia.

[26] Strictly speaking this is correct if the value of v_s^* does not depend on the state of nature. If it does then we need to adjust the formula as we did in the derivation of (13) above.