Lecture 11: Solving DSGE Models Part I
Analytical Solution Methods

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Advanced Monetary Economics 2020

Roadmap

Recap

- Last lecture: we derived the NK Phillips curve.
- Where are we at now with the new Keynesian model?
- All we've really talked about so far is the supply-side of the dynamic model.
- What about the demand-side and government?

Roadmap

Households

- We basically just take the household problem from lecture 7, (the static imperfect competition model), then embed a savings problem.

- Problem:

$$\max_{\{C_t, N_t, B_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi} \right]$$

subject to their budget constraint

$$P_t C_t + Q_t B_{t+1} \leqslant W_t N_t + B_t + D_t$$

where $C_t = \left(\int_0^1 C_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$

Households

- FOCs:

$$\beta^t C_t^{-\sigma} - \lambda_t P_t = 0$$

$$-\beta^t N_t^{\psi} + \lambda_t W_t = 0$$

$$-\lambda_t Q_t + \mathbb{E}_t[\lambda_{t+1}] = 0$$

Log-Linearised Form

- Labour supply and Euler equation (see lecture 4 for derivation):

$$\sigma \hat{c}_t + \psi \hat{n}_t = \hat{w}_t - \hat{p}_t$$

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]).$$

- Recall that we used the labour supply equation in deriving the new Keynesian Phillips curve.

- The Euler equation will form the foundation for the dynamic IS curve.

Log-Linearised Form

- Substitute the market clearing condition ($Y_t = C_t$) into the Euler equation to obtain

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]) \tag{1}$$

- Recall from the last lecture that the output gap is defined as

$$\hat{y}_t^g = \hat{y}_t - \hat{y}_t^n$$

where $\hat{y}_t^n$ is the natural level of output (flexible price equilibrium).

Log-Linearised Form

- Re-write (1) to get the dynamic IS curve

$$\hat{y}_t^g = \mathbb{E}_t[\hat{y}_{t+1}^g] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \hat{r}_t^n)$$

where $\hat{r}_t^n = \sigma \mathbb{E}_t[\hat{y}_{t+1}^n - \hat{y}_t^n]$.

Dynamic IS Curve

- Or expressed differently

$$\hat{y}^g_t = \mathbb{E}_t[\hat{y}^g_{t+1}] - \frac{1}{\sigma}(\hat{r}_t - \hat{r}^n_t)$$

where

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$$
$$\hat{y}^n_t = \mathbb{E}_t[\hat{y}^n_{t+1}] - \frac{1}{\sigma}\hat{r}^n_t$$

- That is — the output gap is proportional to the deviation in the real interest rate from its natural counterpart.

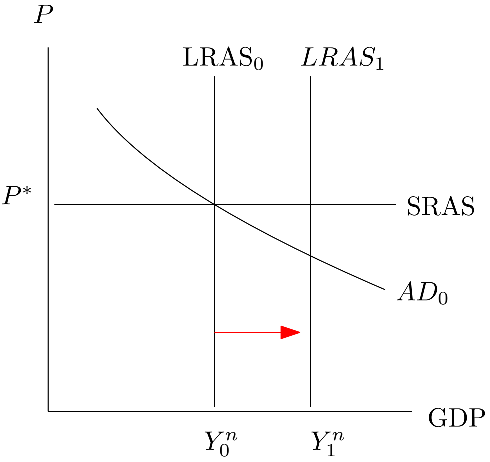
LRAS, Natural Output and Rate of Interest

- How should we interpret this natural rate of interest? From earlier:

$$\hat{y}_t^n = \mathbb{E}_t[\hat{y}_{t+1}^n] - \frac{1}{\sigma}\hat{r}_t^n$$

- It's the real rate of interest that prevails in the flexible price equilibrium.
- All that matters in the flexible price equilibrium is the technology shock a_t.
- Growth in the natural level of output can be interpreted as a rightward shift of the long-run aggregate supply curve.

LRAS, Natural Output and Rate of Interest



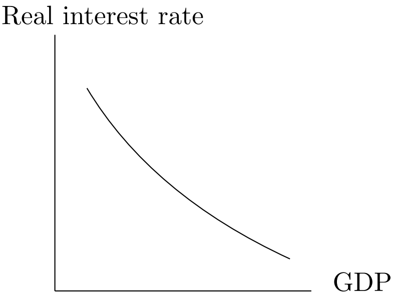
LRAS, Natural Output and Rate of Interest

- Deviations of technology growth that are particularly large $\hat{a}_t > 0$ can be thought of as abnormally large growth in productive capacity.
- Abnormally large shifts in LRAS curve, (booms in the business cycle).
- Technological progress spills-over to affect the real return for the households.

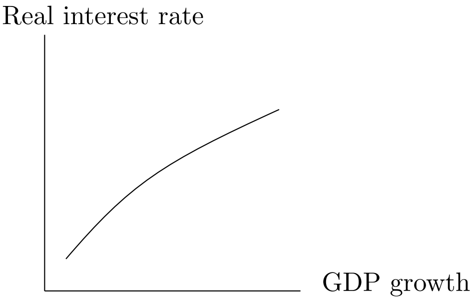
Dynamic IS Curve

- Recall that the traditional Keynesian IS curve plotted output as a function of the interest rate.
- This dynamic analogue looks at the growth rate in output relative to real interest rate.
- Same idea, different presentation.
- Present it in the form of the output gap to relate to the new Keynesian Phillips curve.

Dynamic IS Curve



Traditional IS curve (Keynesian)



Dynamic IS curve (new Keynesian)

Monetary Authority

- Finally, we need to say something about the monetary authority.
- Controls the nominal interest rate: has real impact given rigid prices.
- Taylor rule: a reduced-form way of capturing monetary policy behaviour.

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t^g + \nu_t$$

  where ν_t is a monetary policy shock.

- Responses to both inflation and output gap captures the "dual mandate".

Full System

- Three equations in three unknowns $(\hat{\pi}_t, \hat{i}_t, \hat{y}^g_t)$

$$\hat{\pi}_t = \kappa \hat{y}^g_t + \beta \mathbb{E}_t[\hat{\pi}_{t+1}]$$
$$\hat{y}^g_t = \mathbb{E}_t[\hat{y}^g_{t+1}] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \hat{r}^n_t)$$
$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{y}^g_t + \nu_t$$

- Where exogenous processes are given by

$$\hat{a}_{t+1} = \rho_a \hat{a}_t + \epsilon^a_{t+1}, \ \ \epsilon^a_{t+1} \sim N(0, \sigma^2_a)$$
$$\hat{v}_{t+1} = \rho_v \hat{v}_t + \epsilon^v_{t+1}, \ \ \epsilon^v_{t+1} \sim N(0, \sigma^2_v)$$

- Where to from here?

- How do we solve this?

Roadmap

Solution

- We want to take this three equation system and express it in terms of the shocks in the model?
- How many shocks are in this model?
- Which variables are expressed in terms of which shocks?
- For this model we can find an analytical solution in terms of the parameters of the model using a method of "guess and verify".
- This works in this model since everything is simple.
- Won't necessarily work for bigger and more complicated models, (need numerical methods for this: next lecture).

Guess and Verify

- Guess that the endogenous variables are linearly impacted by shocks.
- Flexible price equilibrium variables are affected by $\hat{a}_t$ only?
- Other variables for the non-flexible price equilibrium also impacted by monetary shocks: due to price rigidities in the short-run.

Guess and Verify

- Conjecture that

$$\hat{y}_t^n = \gamma_{ya}^n \hat{a}_t$$
$$\Rightarrow \hat{r}_t^n = \sigma \gamma_{ya}^n \mathbb{E}_t[\hat{a}_{t+1} - \hat{a}_t]$$

where we knew what γ_{ya}^n was from last lecture. I.e.

$$\hat{y}_t^n = \frac{1+\varphi}{(1-\alpha)\sigma + \alpha + \varphi} \hat{a}_t$$

so $\gamma_{ya}^n = \frac{1+\varphi}{(1-\alpha)\sigma+\alpha+\varphi}$.

Guess and Verify

- Also conjecture that

$$\hat{\pi}_t = \gamma_{\pi a}\hat{a}_t + \gamma_{\pi v}\hat{v}_t$$
$$\hat{y}_t^g = \gamma_{ya}^g\hat{a}_t + \gamma_{yv}^g\hat{v}_t.$$

Guess and Verify

- Iterating forwards then means that

$$\begin{aligned}\mathbb{E}_t[\hat{\pi}_{t+1}] &= \mathbb{E}_t[\gamma_{\pi a}\hat{a}_{t+1} + \gamma_{\pi v}\hat{v}_{t+1}] \\ &= \gamma_{\pi a}\rho_a\hat{a}_t + \gamma_{\pi v}\rho_v\hat{v}_t\end{aligned}$$

$$\begin{aligned}\mathbb{E}_t[\hat{y}^g_{t+1}] &= \mathbb{E}_t[\gamma^g_{ya}\hat{a}_{t+1} + \gamma^g_{yv}\hat{v}_{t+1}] \\ &= \gamma^g_{ya}\rho_a\hat{a}_t + \gamma^g_{yv}\rho_v\hat{v}_t\end{aligned}$$

$$\hat{r}^n_t = \sigma\gamma^n_{ya}(\rho_a - 1)\hat{a}_t$$

what is the intuition behind $\hat{r}^n_t$'s coefficient?

Guess and Verify: NK Phillips Curve

- Substitute these guesses into the new Keynesian Phillips curve

$$
\begin{aligned}
&\hat{\pi}_t = \kappa \hat{y}_t^g + \beta \mathbb{E}_t[\hat{\pi}_{t+1}] \\
&\Rightarrow (\gamma_{\pi a}\hat{a}_t + \gamma_{\pi v}\hat{v}_t) = \kappa(\gamma_{ya}^g \hat{a}_t + \gamma_{yv}^g \hat{v}_t) + \beta(\gamma_{\pi a}\rho_a \hat{a}_t + \gamma_{\pi v}\rho_v \hat{v}_t) \\
&\Rightarrow \hat{a}_t \left\{\kappa \gamma_{ya}^g + \beta \gamma_{\pi a}\rho_a - \gamma_{\pi a}\right\} + \hat{v}_t \left\{\kappa \gamma_{yv}^g + \beta \gamma_{\pi v}\rho_v - \gamma_{\pi v}\right\} = 0
\end{aligned}
$$

notice that the last line must hold for any pair of realisations $(\hat{a}_t, \hat{v}_t)$.

- Follows that

$$
\begin{aligned}
\kappa \gamma_{ya}^g + \beta \gamma_{\pi a}\rho_a - \gamma_{\pi a} &= 0 \\
\kappa \gamma_{yv}^g + \beta \gamma_{\pi v}\rho_v - \gamma_{\pi v} &= 0
\end{aligned}
$$

Guess and Verify: Dynamic IS Curve

- Substitute the monetary rule and $\hat{r}_t^n$ in

$$\hat{y}_t^g = \mathbb{E}_t[\hat{y}_{t+1}^g] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \hat{r}_t^n)$$

$$\Rightarrow \hat{y}_t^g = \mathbb{E}_t[\hat{y}_{t+1}^g] - \frac{1}{\sigma}(\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t^g + \nu_t - \mathbb{E}_t[\hat{\pi}_{t+1}] + \sigma\gamma_{ya}^n(\rho_a - 1)\hat{a}_t)$$

$$\Rightarrow \left\{\gamma_{ya}^g\left(1 + \frac{1}{\sigma}\phi_y - \rho_a\right) + \gamma_{\pi a}\frac{1}{\sigma}\left(\phi_\pi - \rho_a\right) + \gamma_{ya}^n(\rho_a - 1)\right\}\hat{a}_t +$$

$$\left\{\gamma_{yv}^g\left(1 + \frac{1}{\sigma}\phi_y - \rho_v\right) + \gamma_{\pi v}\frac{1}{\sigma}\left(\phi_\pi - \rho_v\right) + 1\right\}\hat{v}_t$$

- Follows that

$$\gamma_{ya}^g\left(1 + \frac{1}{\sigma}\phi_y - \rho_a\right) + \gamma_{\pi a}\frac{1}{\sigma}\left(\phi_\pi - \rho_a\right) + \gamma_{ya}^n(\rho_a - 1) = 0$$

$$\gamma_{yv}^g\left(1 + \frac{1}{\sigma}\phi_y - \rho_v\right) + \gamma_{\pi v}\frac{1}{\sigma}\left(\phi_\pi - \rho_v\right) + 1 = 0$$

Guess and Verify: System of Equations

- Four equations in four unknowns $(\gamma_{\pi v}, \gamma_{\pi a}, \gamma_{yv}^{g}, \gamma_{ya}^{g})$

$$
\begin{aligned}
-(1-\beta\rho_a)\gamma_{\pi a} + \kappa\gamma_{ya}^{g} &= 0 \\
-(1-\beta\rho_v)\gamma_{\pi v} + \kappa\gamma_{yv}^{g} &= 0 \\
\gamma_{\pi a}\frac{1}{\sigma}(\phi_\pi - \rho_a) + \gamma_{ya}^{g}\left(1 + \frac{1}{\sigma}\phi_y - \rho_a\right) &= -\gamma_{ya}^{n}(\rho_a - 1) \\
\gamma_{\pi v}\frac{1}{\sigma}(\phi_\pi - \rho_v) + \gamma_{yv}^{g}\left(1 + \frac{1}{\sigma}\phi_y - \rho_v\right) &= -1
\end{aligned}
$$

Guess and Verify: Matrix Form

$$\begin{bmatrix} 0 & -(1-\beta\rho_a) & 0 & \kappa \\ -(1-\beta\rho_v) & 0 & \kappa & 0 \\ 0 & \frac{1}{\sigma}\left(\phi_\pi - \rho_a\right) & \left(1+\frac{1}{\sigma}\phi_y - \rho_a\right) & 0 \\ \frac{1}{\sigma}\left(\phi_\pi - \rho_v\right) & 0 & \left(1+\frac{1}{\sigma}\phi_y - \rho_v\right) & 0 \end{bmatrix} \begin{bmatrix} \gamma_{\pi v} \\ \gamma_{\pi a} \\ \gamma_{yv}^{g} \\ \gamma_{ya}^{g} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\gamma_{ya}^{n}(\rho_a - 1) \\ -1 \end{bmatrix}$$

- Can in principle invert the matrix of coefficients, multiply by the vector of constants and get the coefficients.
- Can then back-out all the other variables.

Impulse Responses

- The system of variables will respond endogenously to shocks.
- Can trace-out the time paths followed by variables.
- E.g. say a one-time shock to monetary policy and then no further shocks.

$$\begin{aligned}
\hat{\pi}_0 &= \gamma_{\pi v} \hat{v}_0 \\
\hat{\pi}_1 &= \gamma_{\pi v} \hat{v}_1 \\
&= \gamma_{\pi v} [\rho_v \hat{v}_0] \\
\hat{\pi}_1 &= \gamma_{\pi v} \hat{v}_2 \\
&= \gamma_{\pi v} [\rho_v^2 \hat{v}_0] \\
&\dots
\end{aligned}$$

the resulting sequence of variables traced-out is known as an impulse response.

Roadmap

Takeaways

- If the model is simple enough, we can solve it analytically.
- Method of undetermined coefficients.
- If it's not sufficiently simple, we need to use numerical methods — next lecture.