Lecture 14: Theory of Asset Pricing IV
Equilibrium Asset Pricing and Consumption-CAPM

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Roadmap

Motivation

- Let's recap what we've talked about so far.
    - Given prices and potential payoffs, how would a risk averse consumer decide between consumption and a risky asset?
    - How should a risk averse investor allocate their savings amongst several assets? How should investors form portfolios?
    - When do we have an asset price bubble?
- Notice that these are all partial equilibrium questions.
- I.e. given prices, how should investors act?
- How do these prices get determined in equilibrium?

Motivation

- Consumption-based capital asset pricing model (CCAPM).
- What determines the returns on assets in equilibrium.
- All investors are optimising and markets are clearing.

Motivation

- The key for pinning-down the equilibrium prices is market clearing.

Roadmap

Setup

- We return now to our consumption-based asset pricing model.
- We take the problem of the household and close it out with market clearing conditions to price the assets.
- Based on Lucas (1978).

Setup

- Infinite time horizon in discrete time $t \in \{0, 1, 2, ..\}$.
- Household can hold a riskless bond (denote holdings by b_{t+1}) that offers a return of $r_t^F \geq 1$.
- Can also hold a risky asset (denote holdings by a_{t+1}) that offers a dividend stream that's stochastic of $\{d_t\}_{t=0}^{\infty}$.
- Also choose how much to consume.
- Denote the price sequence for the risky asset as $\{p_t\}_{t=0}^{\infty}$.
- The household derives its income from the dividend stream.

Roadmap

Household problem

- Their problem is given by

$$\max_{\{c_t, b_{t+1}, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to

$$c_t + b_{t+1} + p_t a_{t+1} = a_t(p_t + d_t) + r_t^F b_t$$

with b_0 and a_0 taken as given.

Household solution

- Lagrangian given by

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left[a_t(p_t + d_t) + r_t^F b_t - c_t - b_{t+1} - p_t a_{t+1} \right]$$

with first order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = 0 \Rightarrow -\lambda_t + \mathbb{E}_t[r_t^F \lambda_{t+1}] = 0$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow -\lambda_t p_t + \mathbb{E}_t[(p_{t+1} + d_{t+1})\lambda_{t+1}] = 0$$

Household solution

- Gives the Euler equations

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} r_t^F \right]$$

$$p_t = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} [p_{t+1} + d_{t+1}] \right]$$

This is nothing new so far.

Market clearing

- Here is the new part.
- Market equilibrium is defined as a sequence of allocations $\{c_t, a_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ and prices $\{p_t\}_{t=0}^{\infty}$ such that households optimise and all markets clear.
- Market clearing given by
  (1) Goods market: $c_t = d_t$.
  (2) Risky asset market: $a_{t+1} = 1$.
  (3) Riskless bond market: $b_{t+1} = 0$.

Market clearing

- How do we interpret these market clearing conditions?
- Goods: the dividends paid-out by the risky asset are consumption goods.
- There is no production, labour or endowments of any other kind: everything up for potential consumption comes through these dividends.
- Risky asset: it is in unit net supply.
- Means that you can interpret the asset holdings a_{t+1} as holdings of shares in the risky asset.

Market clearing

- Riskless asset: it is in zero net supply.
- You can interpret these as inter-household loans.
- Since there is no government.
- But there is a representative household!
- So prices are adjusting such that the household finds it optimal to not hold any bonds, hold the whole risky asset and consume the whole dividend in equilibrium.

What about heterogeneity?

- What if we relaxed the assumption of a representative household?
- Say we have two households: A and B.
- Denote their optimal choices by $\{c_t^i, b_{t+1}^i, a_{t+1}^i\}_{t=0}^{\infty}$ for $i \in \{A, B\}$.
- How would the market clearing conditions change?

What about heterogeneity?

- Goods: $c_t^A + c_t^B = d_t$.
- Asset: $a_t^A + a_t^B = 1$.
- Riskless bond: $b_{t+1}^A + b_{t+1}^B = 0$.
- Says now that the borrowings through riskless bonds by one household are equal to the savings of the other household through these bonds.
- Note that d_t is the total dividend coming from the riskless asset.
- The risky asset holdings sum to one: the two households have part shares in the asset.

Solving for the prices

- Consumption is the numeraire good here (price is normalised to unity).
- In principle, we can solve for the riskless return and asset prices through plugging the optimal solutions for the household into the market clearing conditions and solving.
- We've priced our assets!

Roadmap

Returns

- What can we say about the relationship between asset returns and consumption smoothing of the household?
- Recall: the household seeks to use these assets to smooth their consumption.
- Sell assets when times are bad and buy assets when times are good.

Returns

- Recall that we can write our Euler equation for the risky asset in terms of returns as follows

$$p_t = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} [p_{t+1} + d_{t+1}] \right]$$

$$\Rightarrow 1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{[p_{t+1} + d_{t+1}]}{p_t} \right]$$

$$\Rightarrow 1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} r_{t+1} \right]$$

Stats

- Remember back to your statistics classes...
- $\mathbb{E}[xy] = \mathbb{E}[x]\mathbb{E}[y] + Cov(x, y)$ for any two random variables x and y.
- The stochastic discount factor $\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}$ here and r_{t+1} are both random variables!

Returns

- Hence we can write the right-side of the Euler equation as

$$
\begin{aligned}
1 &= \mathbb{E}_t\left[\mathcal{M}_{t+1} r_{t+1}\right] \\
1 &= \mathbb{E}_t[\mathcal{M}_{t+1}]\mathbb{E}_t[r_{t+1}] + Cov(\mathcal{M}_{t+1}, r_{t+1})
\end{aligned}
$$

where I've re-written the SDF as $\mathcal{M}_{t+1}$ for ease of notation.

Returns

- Recall from the consumption asset pricing lecture that

$$\mathbb{E}_t[\mathcal{M}_{t+1}] = \frac{1}{r^F_{t+1}}$$

i.e. the expected SDF equals the reciprocal of the riskless rate.

- Hence

$$\begin{aligned} 1 &= \mathbb{E}_t\left[\frac{r_{t+1}}{r^F_{t+1}}\right] + Cov(\mathcal{M}_{t+1}, r_{t+1}) \\ \Rightarrow \mathbb{E}_t[r_{t+1}] &= r^F_{t+1}\{1 - Cov(\mathcal{M}_{t+1}, r_{t+1})\} \\ \Rightarrow \mathbb{E}_t[r_{t+1}] - r^F_{t+1} &= -r^F_{t+1} Cov(\mathcal{M}_{t+1}, r_{t+1}) \end{aligned}$$

where the left-side is excess return of the risky asset over the riskless one.

Returns

- Can we go any further here?

$$\mathbb{E}_t[r_{t+1}] - r^F_{t+1} = -r^F_{t+1} Cov(\mathcal{M}_{t+1}, r_{t+1}) = -\frac{1}{\mathbb{E}_t[\mathcal{M}_{t+1}]} Cov(\mathcal{M}_{t+1}, r_{t+1})$$

where recall that

$$\mathbb{E}_t[\mathcal{M}_{t+1}] = \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right] = \left(\frac{1}{c_t}\right)^{-\sigma}\mathbb{E}_t\left[\beta\left(c_{t+1}\right)^{-\sigma}\right]$$

given that c_t is known at time t.

Returns

- See then that

$$\begin{aligned}
\mathbb{E}_t[r_{t+1}] - r^F_{t+1} &= -\frac{1}{\mathbb{E}_t[\mathcal{M}_{t+1}]} Cov(\mathcal{M}_{t+1}, r_{t+1}) \\
&= -\left(\frac{1}{c_t}\right)^{\sigma} \frac{1}{\mathbb{E}_t\left[\beta\left(c_{t+1}\right)^{-\sigma}\right]} \\
&\quad Cov\left(\left(\frac{1}{c_t}\right)^{-\sigma}\left[\beta\left(c_{t+1}\right)^{-\sigma}\right], r_{t+1}\right)
\end{aligned}$$

Returns

- The β and c_t terms from the expectation and covariance cancel to give

$$\mathbb{E}_t[r_{t+1}] - r^F_{t+1} = -\frac{1}{\mathbb{E}_t\left[(c_{t+1})^{-\sigma}\right]} Cov((c_{t+1})^{-\sigma}, r_{t+1})$$

which says that the excess return on a risky asset depends on its co-movements with consumption.

Returns

- The expression says that if the covariance between r_{t+1} and the marginal utility of consumption is positive, then the excess return is negative.
- Recall that a decrease in consumption is what gives an increase in the marginal utility of consumption.
- An asset that pays off when consumption is low has a negative excess return.
- Returns that move against consumption are a hedge against consumption risk: investors are willing to accept a lower expected return.

Roadmap

Summary

- Equilibrium asset pricing involves adding market clearing conditions to the basic asset pricing model.
- We can then obtain prices from a set of equations.
- Basic manipulations of the Euler equation gives the consumption CAPM: the prediction that covariance with consumption is what determines an asset's expected return.