Lecture 1: Introduction and Mathematical Methods

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Advanced Financial Economics 2019

Roadmap

Instructor

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- Ph.D. in Economics, Ph.D. in Finance, M.S. Economics.
  - University of Wisconsin-Madison (USA).
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Course Overview

- The course is split into four parts.
  (1) Corporate finance.
  (2) Asset pricing.
  (3) Financial intermediation.
  (4) Macro-finance.

(1) Corporate finance

- Say a firm wants to take a new project.
- Corporate finance asks the question of how they best finance the new project?
- In this part of the course, we seek to answer two questions:
  - (A) What does theory predict the optimal financing mix should be?
  - (B) Which factors matter most quantitatively (in the data)?

(2) Asset pricing

- Here we consider
  - (A) How do we characterise asset prices using microfoundations?
  - (B) When, in the real world, can we say that there is an asset price bubble?

(3) Financial intermediation

- Here we ask the question
  (A) Why are there financial intermediaries? What purpose do they serve? What value do they create?

(4) Macro-Finance

- Here we ask the question

  (A) What are the problems and potential remedies associated with a financial intermediation system?

Tools

- This course covers a lot of ground and thus requires a lot of different tools.
- We'll make theoretical predictions, test them with data and leverage both simultaneously using structural models.

Summary

- The material covered in this course will be tough!
- It will be MATHEMATICAL IN NATURE.
- You'll get exposure to lots of new things: may seem intimidating.
- Look through all the math to see the intuition of models and solutions.
- This is not a math course!

Roadmap

Shareholders

- A firm has many stakeholders.
- E.g. creditors, shareholders, managers, workers, etc.
- In this course, we'll assume that a firm is run by a manager.
- Sometimes the manager will be self-interested (we refer to these as agency conflicts).
- But generally, we'll assume that a firm's objective is to maximise the expected discounted value of cash flows that it pays to its shareholders.
- Shareholders are the owners of the firm.

Creditors v.s. Shareholders

- If a firm borrows money, the agent who loans the money is called a creditor or a debtholder.
- The firm is obliged to repay the amount it owes its creditors always.
- If cashflows are insufficient to do so, the firm defaults/goes bankrupt.
- The firm's shareholders invest some initial funds in the firm and then receive dividends in the future.
- Shareholders are protected by limited liability: means that if the firm goes bankrupt, nobody can ask the shareholders for more money.

Roadmap

NPV Analysis

- When evaluating a project, we use NPV analysis.
- Takes the cash flows associated with the project, discounts them and adds them together.
- Why is discounting necessary?
- Opportunity cost of time: instead of investing in this project, I could take the funds and stick them into a riskless bank account and get interest earnings.
- Interest earnings are the opportunity cost.

Discount rate and discount factor

- What's the appropriate opportunity cost to use?
- Depends on our investors.
- What is their opportunity cost? What is their discount factor?
- Assume that time in the world is discrete $t \in \{0, 1, 2, 3, ...\}$.
- If their net opportunity cost is $r > 0$ per period ($1 + r$ gross return), then how to we discount cash flows for them?

Discount rate and discount factor

- I.e. what is £1 received at $t+1$ worth in terms of time t money?
- Invest £1 in this bank account at $t \Rightarrow$ get back £(1+r) at $t+1$. We seek x in the following

$$1 \text{ at } t \Rightarrow (1+r) \text{ at } t+1$$
$$x \text{ at } t \Rightarrow 1 \text{ at } t+1$$

  which yields $x = \frac{1}{1+r}$.
- I.e. £1 received at $t+1$ is worth $\frac{1}{1+r}$ at time t.
- The sooner we get money, the better! We can do more with it!
- Object r is referred to as the discount rate.
- An object defined as $\beta = \frac{1}{1+r}$ is referred to as the discount factor.

Example

- E.g. consider a project that Firm A is contemplating taking. The project has an upfront cost of $c_0 > 0$ and then generates $c > 0$ in positive cash flows in perpetuity from $t = 0$ onwards. The investors in the firm can invest in a riskless bank account that offers $r > 0$ of net interest per period. What is the NPV of this project?

$$
\begin{aligned}
NPV &= -c_0 + c + \frac{1}{1+r}c + \left(\frac{1}{1+r}\right)^2 c + \left(\frac{1}{1+r}\right)^3 c + \dots \\
&= -c_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c \\
&= -c_0 + \sum_{t=0}^{\infty} \beta^t c \\
&= -c_0 + \frac{c}{1-\beta}
\end{aligned}
$$

where the penultimate line follows from the definition of the discount factor and the last line comes from geometric series.

Example

- E.g. when will the example on the previous slide constitute a good project from the perspective of the investors? When the NPV is positive!

$$-c_0 + \frac{c}{1-\beta} \geq 0$$
$$\Rightarrow c_0 \leq \frac{c}{1-\beta}$$
$$\Rightarrow \beta \geq \frac{c}{c_0} - 1$$

what does this mean?

- Says that the project is good only if the investors are sufficiently patient. Make sense?
- Notice that this evaluates the project relative to our next best alternative.

NPV versus utility

- This symbol β hopefully looks somewhat familiar in this context.

- Recall lifetime utility is often given as

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

  where $u(c_t)$ is referred to as the period utility function and c_t is consumption for period t.

- A risk averse household has $u(c_t)$ concave in consumption, (i.e. $u' > 0$ and $u'' < 0$).

- A risk neutral household has $u(c_t)$ as being linear in consumption.

NPV versus utility

- Say there is no upfront cost for a project.
- And that it pays out a sequence $\{c_t\}_{t=0}^{\infty}$

$$NPV = \sum_{t=0}^{\infty} \beta^t c_t$$

which says that NPV is the utility associated with consuming the cash flows for a risk-neutral investor.

- In corporate finance, we'll typically assume risk neutral investors.
- Not the case in asset pricing though (more on this later in the semester).

Roadmap

Constrained Optimisation

- "Economics is the study of how society manages its scarce resources" (Mankiw, 2007, *Principles of Economics*).
- Constrained optimisation!

Discrete Time Deterministic Program

- Consider a problem of the form

$$\max_{\vec{x}_t} \sum_{t=0}^{\infty} f(\vec{x}_t, p, t) \text{ s.t. } g(\vec{x}_t, p, t) = \gamma_t \ \forall t \geq 0$$

- Has the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} f(\vec{x}_t, p, t) + \sum_{t=0}^{\infty} \lambda_t[\gamma_t - g(\vec{x}_t, p, t)]$$

where $\lambda_t \geq 0$ are the Lagrange multipliers.

Dynamic Optimisation Example

- Solve the following program for a risk averse household

$$\max_{\{c_t, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

for $\beta \in [0,1]$ subject to the constraint

$$c_t + q_t b_{t+1} = b_t + y_t$$

and with b_0 given. See that b_t are discount bonds, y_t is their income endowment, $q_t < 1$ is the bond price and the price sequence $\{q_t\}_{t=0}^{\infty}$ is taken as given.

- Notice that the dynamics have an effect through savings, b_t.

- What are the control variables here?

Dynamic Optimisation Example Solution (1)

- Lagrangian given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t [b_t + y_t - c_t - q_t b_{t+1}]$$

which comes with first order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - \lambda_t = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial b_t} = 0 \Rightarrow -q_t \lambda_t + \lambda_{t+1} = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow c_t + q_t b_{t+1} = b_t + y_t, \quad (3)$$

Discrete Time Optimisation Example Solution (2)

- Combining (1) and (2) yields

$$\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} = q_t \tag{4}$$

  which is referred to as a consumption Euler equation.

- Equations (3) and (4) together summarise the solution to the program.

Roadmap

Constraints

- Probably for most problems you've seen so far using Lagrangians, we always have the constraint binding.
- E.g. the budget constraint always binds due to always positive marginal utility of consumption.
- Not always the case though.
- E.g. a credit card limit is a constraint.
- You don't necessarily want to borrow right-up to your limit as this can make your financial situation worse next month.

Complementarity Slackness

- Come back to the problem of the form

$$\max_{\vec{x}_t} \sum_{t=0}^{\infty} f(\vec{x}_t, p, t) \text{ s.t. } g(\vec{x}_t, p, t) = \gamma_t \ \forall t \geq 0$$

- Has the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} f(\vec{x}_t, p, t) + \sum_{t=0}^{\infty} \lambda_t[\gamma_t - g(\vec{x}_t, p, t)]$$

where $\lambda_t \geq 0$ are the Lagrange multipliers.

Complementarity Slackness

- With the constraints, we're used to having $\lambda_t > 0$, meaning that $\gamma_t = g(\vec{x}_t, p, t)$.
- In general though, it's the case that

$$\lambda_t[\gamma_t - g(\vec{x}_t, p, t)] = 0$$

  meaning that if $\gamma_t > g(\vec{x}_t, p, t)$, then $\lambda_t = 0$.

- That's all Kuhn-Tucker says.

Complementarity Slackness

- To re-iterate, if a constraint is slack then the Lagrange multiplier equals zero.
- If the constraint binds, then the multiplier is positive, (the case we're used to with budget constraints).

Roadmap

Shocks

- The examples we've looked at so far were all deterministic.
- What happens when we add random shocks to the model?
- Control variables will be a function of realised state of the world.

Randomness and States of Nature

- In this course, we'll assume that there is an information set that evolves over time denoted by $\mathcal{I}_t$.
- In the future, there is some set of possible outcomes $\omega_i \in \Omega$.
- All the agents in the model know the set Ω for the future, they just don't know what ω_i will come up.
- Take expectations over the states and form state-contingent plans for control variables.
- $\mathbb{E}_t[x]$ is shorthand for $\mathbb{E}[x|\mathcal{I}_t]$

Two Period Stochastic Model Example

- Consider an optimal savings problem for a consumer over two periods $t \in \{0, 1\}$.
- The consumer receives endowment of income y_t in period t where $y_t = \bar{y} + \epsilon_t$ where $\mathbb{E}[\epsilon_t] = 0$.
- Consumer maximises NPV of expected lifetime utility where period utility function is $\frac{c_t^{1-\sigma}}{1-\sigma}$.
- Assume that price of consumption in each period is unity and bond price is fixed at q_0.
- Variables will all be functions of the state realised at decision time $\omega_t \in \Omega$.

Two Period Stochastic Model Example

- The consumer is faced with the problem:

$$\max_{c_0(\omega_0), c_1(\omega_1), b_0(\omega_0)} \mathbb{E}_0 \left[\frac{c_0((\omega_0))^{1-\sigma}}{1-\sigma} + \beta \frac{c_1((\omega_1))^{1-\sigma}}{1-\sigma} \right]$$

subject to

$$c_0(\omega_0) + q_0 b_0(\omega_0) = y_0(\omega_0)$$
$$c_1(\omega_1) = b_0(\omega_0) + y_1(\omega_1)$$

Two Period Stochastic Model Example Solution

- Objective given by,

$$\mathcal{L} = \mathbb{E}_0 \left[\frac{(y_0(\omega_0) - q_0 b_0(\omega_0))^{1-\sigma}}{1-\sigma} + \beta \frac{(b_0(\omega_0) + y_1(\omega_1))^{1-\sigma}}{1-\sigma} \right]$$

  which is a function of only one control b_0 from substituting out c_0 and c_1.

- Optimality condition given by

$$\frac{d\mathcal{L}}{db_0} = 0 \Rightarrow q_0 c_0(\omega_0)^{-\sigma} = \beta \mathbb{E}_0[c_1^{-\sigma}(\omega_1)]$$

  which is a stochastic consumption Euler equation.

- See that the optimal decision depends on the state realised at $t = 0$ and what's expected at $t = 1$.

Roadmap

Games

- In the most basic of economic models, we assume that agents are all taking market behaviour as given.
- E.g. perfect competition.
- What if we have a small number of people interacting with each other?
- It will often make sense to think that they take others' behaviour into account.

Nash Equilibrium

- The solution concept we'll use for strategic games is Nash equilibrium.
- We're in a Nash equilibrium if no player in a game can improve their payoff through changing their actions.

Nash equilibrium example

- E.g. find the Nash equilibrium of the following strategic game of two players.

P1/P2	I	II
A	1, -1	2, 1
B	0, 1	4, 2

  where the row player is denoted as P1 and the column player is named P2. P1 has strategy set $\{A, B\}$ and P2 has $\{I, II\}$.

- The payoffs (think of these as utils) of P1 are written first and that for P2 are written second.

- What is each player's best response given fixed behaviour of the other player?

Nash equilibrium example solution

- If P2 chooses I $\Rightarrow$ P1 should play A.
- If P2 chooses II $\Rightarrow$ P1 should play B.
- If P1 chooses A $\Rightarrow$ P2 should play II.
- If P1 chooses B $\Rightarrow$ P2 should play II.
- Where is there overlap? (B, II) is the unique pure strategy Nash equilibrium since neither player can do better by changing their strategy.

Roadmap

Topics Covered

- These mathematical techniques are just tools.
- If you understand how to implement all these methods today, you'll be good for the basic techniques needed for this module.