Lecture 1: Mathematical Methods I

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Roadmap

Instructor

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  - No need for formalities: call me either Adam or Spencer.
- Assistant Professor of Economics (started here this September).
- Ph.D. Economics and Finance, M.S. Economics.
  - University of Wisconsin-Madison (USA).
- M.Econ. (Hons), B.Comm. (Hons) Economics.
  - The University of Melbourne (Australia).

Summary

- The material covered in this course will be tough!
- You'll get exposure to lots of new things: may seem intimidating.
- Look through all the math to see the intuition of models and solutions.
- This is not a math course!

Note

- These mathematical methods are just recipes that I want you to know how to use.
- Again, this is not a math course: these are just tools for doing economics.

Roadmap

Constrained Optimisation

- "Economics is the study of how society manages its scarce resources" (Mankiw, 2007, *Principles of Economics*).
- Constrained optimisation!

Static Program

- A static optimisation program will have the following general form

$$\max_{\vec{x}} \; f(\vec{x}, u) \text{ s.t. } g(\vec{x}, u) = \gamma$$

where $\vec{x}$ is a vector of control variables and u are parameters.

- This will have the following Lagrangian

$$\mathcal{L} = f(\vec{x}, u) + \lambda[\gamma - g(\vec{x}, u)]$$

where $\lambda \geq 0$ is called the Lagrange multiplier.

Static Program

- Interior solution found by taking $\frac{\partial \mathcal{L}}{\partial x_i}$ for all $x_i \in \vec{x}$ and $\frac{\partial \mathcal{L}}{\partial \lambda}$ and equating the derivatives with zero (first order conditions).

- We'll focus just on interior solutions, (corner solutions require the use of Kuhn-Tucker conditions).

Static Optimisation Example

- Solve the following consumption-leisure tradeoff problem:

$$\max_{c,n} \; \frac{c^{1-\sigma}}{1-\sigma} - \chi n$$

subject to $c = wn$ where w is taken as given.

Static Optimisation Example Solution (1)

- Lagrangian given by

$$\mathcal{L} = \frac{c^{1-\sigma}}{1-\sigma} - \chi n + \lambda[wn - c]$$

- First order conditions (FOCs) given by

$$\frac{\partial \mathcal{L}}{\partial c} = 0 \Rightarrow c^{-\sigma} - \lambda = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial n} = 0 \Rightarrow -\chi + \lambda w = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow wn - c = 0 \quad (3)$$

Static Optimisation Example Solution (2)

- Equations (1) and (2) imply

$$c^{-\sigma} = \frac{\chi}{w} \Rightarrow c = \left(\frac{\chi}{w}\right)^{-\frac{1}{\sigma}} \tag{4}$$

- Plug (4) into (3) to get the solution for n as

$$n = \left(\frac{\chi}{w}\right)^{-\frac{1}{\sigma}} / w$$

Roadmap

Discrete Time Deterministic Program

- Consider a problem of the form

$$\max_{\vec{x}_t} \sum_{t=0}^{\infty} f(\vec{x}_t, u, t) \text{ s.t. } g(\vec{x}_t, u, t) = \gamma_t \;\; \forall t \geq 0$$

where notice the time subscripts now. Why none on u?

- Has the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} f(\vec{x}_t, u, t) + \sum_{t=0}^{\infty} \lambda_t [\gamma_t - g(\vec{x}_t, u, t)]$$

where $\lambda_t \geq 0$ are the Lagrange multipliers.

Discrete Time Optimisation Example

- Solve the following program

$$\max_{\{c_t, n_t, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi n_t \right]$$

for $\beta \in [0, 1]$ subject to the constraint

$$p_t c_t + q_t b_t = b_{t-1} + w_t n_t$$

where b_t are discount bonds, $(q_t < 1)$ and the price sequences $\{w_t, p_t, q_t\}_{t=0}^{\infty}$ are taken as given.

- Notice that the dynamics have an effect through savings, b_t.

Discrete Time Optimisation Example Solution (1)

- Lagrangian given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi n_t \right] + \sum_{t=0}^{\infty} \lambda_t [b_{t-1} + w_t n_t - p_t c_t - q_t b_t]$$

which comes with first order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - \lambda_t p_t = 0 \quad (5)$$
$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Rightarrow -\beta^t \chi + \lambda_t w_t = 0 \quad (6)$$
$$\frac{\partial \mathcal{L}}{\partial b_t} = 0 \Rightarrow -q_t \lambda_t + \lambda_{t+1} = 0 \quad (7)$$
$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow p_t c_t + q_t b_t = b_{t-1} + w_t n_t, \quad (8)$$

Discrete Time Optimisation Example Solution (2)

- Recall that the price sequences are all taken as given (exogenous).
- Using (5) and (6) yields

$$c_t^{-\sigma} = \frac{\chi}{w_t/p_t} \Rightarrow c_t = \left(\frac{\chi}{w_t/p_t}\right)^{-\frac{1}{\sigma}} \tag{9}$$

- FOC (5) tells us that

$$\lambda_t = \frac{\beta^t c_t^{-\sigma}}{p_t} \tag{10}$$

Discrete Time Optimisation Example Solution (3)

- Combining (7) and (10) yields

$$q_t = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{p_t}{p_{t+1}} \tag{11}$$

  which is referred to as a consumption Euler equation.

- Equations (11), (9) and (8) together summarise the solution to the program.

Discrete Time Optimisation Example Solution (4)

- Solution to the program is given by a sequence $\{c_t, n_t, b_t\}_{t=0}^{\infty}$ that satisfies

$$q_t = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{p_t}{p_{t+1}}$$

$$c_t = \left(\frac{w_t}{p_t} \frac{1}{\chi} \right)^{\sigma}$$

$$p_t c_t + q_t b_t = b_{t-1} + w_t n_t$$

together with initial condition b_{-1} and "no ponzi game" restriction

$$\lim_{t \to \infty} \left[\prod_{j=0}^{t} q_j \right] b_t = 0$$

which says that the NPV of the "terminal" asset holdings are zero.

Roadmap

Shocks

- The examples we've looked at so far were all deterministic.
- What happens when we add random shocks to the model?
- Control variables will be a function of realised state of the world.

Randomness and States of Nature

- In this course, we'll assume that there is an information set that evolves over time denoted by $\mathcal{I}_t$.
- In the future, there is some set of possible outcomes $\omega_i \in \Omega$.
- All the agents in the model know the set Ω for the future, they just don't know what ω_i will come up.
- Take expectations over the states and form state-contingent plans for control variables.
- $\mathbb{E}_t[x]$ is shorthand for $\mathbb{E}[x|\mathcal{I}_t]$

Two Period Stochastic Model Example

- Consider an optimal savings problem for a consumer over two periods $t \in \{0, 1\}$.
- The consumer receives endowment of income y_t in period t where $y_t = \bar{y} + \epsilon_t$ where $\mathbb{E}[\epsilon_t] = 0$.
- Consumer maximises NPV of expected lifetime utility where period utility function is $\frac{c_t^{1-\sigma}}{1-\sigma}$.
- Assume that price of consumption in each period is unity and bond price is fixed at q_0.
- Variables will all be functions of the state realised at decision time $\omega_t \in \Omega$.

Two Period Stochastic Model Example

- The consumer is faced with the problem:

$$\max_{c_0(\omega_0), c_1(\omega_1), b_0(\omega_0)} \quad \mathbb{E}_0 \left[\frac{c_0((\omega_0))^{1-\sigma}}{1-\sigma} + \beta \frac{c_1((\omega_1))^{1-\sigma}}{1-\sigma} \right]$$

subject to

$$c_0(\omega_0) + q_0 b_0(\omega_0) = y_0(\omega_0)$$
$$c_1(\omega_1) = b_0(\omega_0) + y_1(\omega_1)$$

Two Period Stochastic Model Example Solution

- Objective given by,

$$\mathcal{L} = \mathbb{E}_0 \left[\frac{(y_0(\omega_0) - q_0 b_0(\omega_0))^{1-\sigma}}{1-\sigma} + \beta \frac{(b_0(\omega_0) + y_1(\omega_1))^{1-\sigma}}{1-\sigma} \right]$$

which is a function of only one control b_0 from substituting out c_0 and c_1.

- Optimality condition given by

$$\frac{d\mathcal{L}}{db_0} = 0 \Rightarrow q_0 c_0(\omega_0)^{-\sigma} = \beta \mathbb{E}_0[c_1^{-\sigma}(\omega_1)]$$

which is a stochastic consumption Euler equation.

- See that the optimal decision depends on the state realised at $t = 0$ and what's expected at $t = 1$.

Two Period Stochastic Model Example Solution

- Can we solve $q_0 c_0(\omega_0)^{-\sigma} = \beta \mathbb{E}_0[c_1^{-\sigma}(\omega_1)]$ for $b_0(\omega_0)$ in closed form?
- No! Either use numerical methods or local approximations.
- As is canonical in monetary economics, we'll use lots of local approximations through the log-linearisation technique.
- Note: from now on, I'll drop the state scripts to ease notation (i.e. y_0 rather than $y_0(\omega_0)$).

Roadmap

Steady State

- We reach a steady state when nothing is changing.
- In the previous example, this is given by $\epsilon_t = 0$ for each period.
- That is: $y_0 = y_1 = \bar{y}$.
- Other variables will be unchanging as well $c_0 = c_1 = \bar{c}$.

Steady State Example

- In the steady state of the two-period model, see

$$q_0 \bar{c}^{-\sigma} = \beta \bar{c}^{-\sigma} \Rightarrow q_0 = \beta \quad (12)$$
$$\bar{c} = \bar{y} - q_0 b_0 \quad (13)$$
$$\bar{c} = \bar{y} + \bar{b}_0. \quad (14)$$

- For (12) – (14) to all hold, we need for no savings (i.e. $b_0 = 0$) between periods.

- Follows that $\bar{c} = \bar{y}$: consumption each period just equals the deterministic endowment.

Log-Linearisation

- Approximates non-linear solutions around the steady state.
- Define the log deviation of a variable (x_t) from its steady state as

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right)$$

Log-Linearisation

- We can interpret $\hat{x}_t$ as a percentage deviation of the variable from its steady state as:

$$\begin{aligned}\hat{x}_t &= \log\left(\frac{x_t}{\bar{x}}\right) \\ &= \log\left(1 + \frac{x_t - \bar{x}}{\bar{x}}\right) \\ &= \frac{x_t - \bar{x}}{\bar{x}} + \text{higher order terms}\end{aligned}$$

where the third line is a Taylor expansion.

- I.e. $\log(1 + y) \approx y + \text{higher order terms}$.

Log-Linearisation

- A first order Taylor expansion drops these higher-order terms.
- So at a first order, we can approximate

$$\hat{x}_t = \frac{x_t - \bar{x}}{\bar{x}},$$

which says that $\hat{x}_t$ is approximately a percentage deviation about the steady state $\bar{x}$.

Log-Linearisation Example

- Linearise the consumption Euler equation from the two period model:

$$q_0 c_0^{-\sigma} = \beta \mathbb{E}_0[c_1^{-\sigma}]$$

- Assume that q_0 is a fixed parameter, as is β.
- c_0 and c_1 are endogenous and can deviate though.

Log-Linearisation Example Solution

- From our definition of the deviations, see that

$$\hat{c}_0 = \log\left(\frac{c_0}{\bar{c}}\right) \Rightarrow c_0 = \bar{c}e^{\hat{c}_0}.$$

- We can plug this into the Euler equation to get

$$\begin{aligned} q_0(\bar{c}e^{\hat{c}_0})^{-\sigma} &= \beta\mathbb{E}_0[(\bar{c}e^{\hat{c}_1})^{-\sigma}] \\ \Rightarrow q_0\bar{c}^{-\sigma}e^{-\sigma\hat{c}_0} &= \beta\bar{c}^{-\sigma}\mathbb{E}_0[e^{-\sigma\hat{c}_1}] \\ \Rightarrow e^{-\sigma\hat{c}_0} &= \mathbb{E}_0[e^{-\sigma\hat{c}_1}] \\ \Rightarrow (1-\sigma\hat{c}_0) &= \mathbb{E}_0[(1-\sigma\hat{c}_1)] \\ \Rightarrow \hat{c}_0 &= \mathbb{E}_0[\hat{c}_1] \end{aligned}$$

where the third line comes from steady state equation (12) and the fourth line comes from a first order Taylor expansion of $\exp(1+x)$.

Roadmap

Topics Covered

- These mathematical techniques are just tools.
- If you understand how to implement all these methods today, you'll be good for the math in this first half of the course.