

Lecture II

Solving Representative Agent General Equilibrium Models

Adam Hal Spencer

The University of Nottingham

Applied Computational Economics 2020

Roadmap

- 1 Introduction
- 2 Equilibrium Concepts in Dynamic Models
- 3 Extending to Stochastic Models
- 4 Computing RCE via Value Function Iteration
- 5 Transition Dynamics Between RCE: Shooting Algorithm
- 6 Local Solutions: Perturbation Methods
- 7 Conclusion

Outline

- We've gotten the basic stuff out of the way.
- From here we go:
 - Representative agents in general equilibrium,
 - Heterogeneous agents, (idiosyncratic uncertainty),
 - Heterogeneous agents, (aggregate uncertainty).

Equilibrium Concepts

- How do we define a **dynamic** general equilibrium?
- In a static context, we study general equilibrium with a finite number of goods.
- When we start talking about dynamics, we get into the realm of infinite-dimensional spaces...

Roadmap

- 1 Introduction
- 2 Equilibrium Concepts in Dynamic Models**
- 3 Extending to Stochastic Models
- 4 Computing RCE via Value Function Iteration
- 5 Transition Dynamics Between RCE: Shooting Algorithm
- 6 Local Solutions: Perturbation Methods
- 7 Conclusion

Equilibrium Concepts

- Three predominant approaches we can take when dealing with market economies, (i.e. not social planner's problem)
 - Arrow-Debreu (valuation) equilibrium,
 - Sequential markets equilibrium,
 - Recursive competitive equilibrium.
- The third is the most useful from a **computational** perspective.
- These types of decentralised equilibria are interesting when we start introducing distortions, (e.g. taxes).
- Can no longer just use the social planner's problem.
- Let's briefly contrast these three equilibrium concepts.

Neoclassical General Equilibrium Model

- Consider the **deterministic market** version of our favourite problem (where the household owns the capital stock)
- Household's problem is

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} k_{t+1} + c_t &= r_t k_t + (1 - \delta)k_t + \pi_t \\ c_t, k_{t+1} &\geq 0 \end{aligned}$$

with k_0 .

- Firm's problem:

$$\max_{\{k_t\}} \pi_t = k_t^\alpha - r_t k_t$$

- In this setup, consumption **at time t** is the numeraire.

Concept (1): Arrow-Debreu Equilibrium

- An A-D equilibrium treats this as a static GE problem with an infinite number of goods to allocate across (given that we have infinite time periods).
- The households all trade **only at $t = 0$** and deal in irrevocable claims to commodities indexed by time.
- They buy claims to consumption at any arbitrary time $t \geq 0$.
- Claim trading closes at the end of $t = 0$. These markets then **close forever** and then the “world plays-out” until the end of time.
- Firms produce and hand-over their goods, but only in accordance with what was agreed at $t = 0$, (no more negotiations).
- All agents bound to follow contracts set-out at $t = 0$.

Concept (1): Arrow-Debreu Equilibrium

- We take consumption at time $t = 0$ as the numeraire here.
- Price of consumption at time t relative to time $t = 0$ is denoted p_t , (where $p_0 = 1$).

Concept (1): Arrow-Debreu (A-D) Equilibrium

- An A-D equilibrium is a set of
 - Prices $\{p_t^*\}_{t=0}^{\infty}$ (consumption goods), $\{r_t^*\}_{t=0}^{\infty}$ (rental rate on capital),
 - Quantities $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$, such that

(1) Sequence $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solve the consumer's problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to their time $t = 0$ budget constraint

$$\sum_{t=0}^{\infty} p_t^* [c_t + k_{t+1}] \leq \sum_{t=0}^{\infty} p_t^* [r_t^* k_t + (1 - \delta)k_t]$$

with k_0 taken as given.

- Notice that the price r_t^* is relative to time t consumption goods.

Concept (1): Arrow-Debreu (A-D) Equilibrium

(2) Sequence $\{k_t^*\}_{t=0}^{\infty}$ solves the firm problem

$$\max_{\{k_t\}} p_t^* k_t^\alpha - p_t^* r_t^* k_t$$

(3) Markets clear

$$c_t^* + k_{t+1}^* = (k_t^*)^\alpha + (1 - \delta)k_t^*$$

Concept (2): Sequential Markets Equilibrium

- How about we have trading in assets taking place at every $t \geq 0$.
- Each period, markets open all the trading takes place, they close and then open again next period.
- Seems a bit more natural...

Concept (2): Sequence Markets Equilibrium

- A sequential markets equilibrium is a set of
 - Prices $\{r_t^*\}_{t=0}^{\infty}$ (rental rate on capital),
 - Quantities $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$, such that

(1') Sequence $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solve the consumer's problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to their **time t** budget constraint

$$c_t + k_{t+1} = r_t^* k_t + (1 - \delta)k_t$$

with k_0 taken as given.

- Notice that the price r_t^* is relative to time t consumption goods.

Concept (2): Sequence Markets Equilibrium

(2') Sequence $\{k_t^*\}_{t=0}^{\infty}$ solves the firm problem

$$\max_{\{k_t\}} k_t^\alpha - r_t^* k_t$$

$\forall t.$

(3') Markets clear

$$c_t^* + k_{t+1}^* = (k_t^*)^\alpha + (1 - \delta)k_t^*$$

$\forall t.$

A-D and Sequential Markets Equilibria

- Notice again that these two methods involve computing **infinite sequences**.
- Again, computers don't like that.
- Remember we had this same problem with the household's sequence problem?
- Our solution there was the recursive formulation.
- Can something similar save us here?

Concept (3) Recursive Competitive Equilibrium

- We can start by again thinking about the household's recursive formulation.
- There's a slight twist on what we looked at in the partial equilibrium setup last time.
- Factor prices in this market economy are functions of the representative agents' choices, but they're meant to be taking these things as **given**.
- Trick: "big-K, little-k".

Concept (3) Recursive Competitive Equilibrium

- The household's recursive formulation is given by

$$v(k, K) = \max_{\{c, k'\}} u(c) + \beta v(k', K')$$

subject to

$$\begin{aligned}c + k' &= R(K)k + (1 - \delta)k \\ K' &= G(K)\end{aligned}$$

where notice that the household now has two states — k and K .

- $R(K)$ is the rental rate on capital: determined at the aggregate level from production function K^α .
- The second constraint is the law of motion of aggregate capital, which the household takes as given.

Concept (3) Recursive Competitive Equilibrium

- k denotes the household's current capital stock.
- K denotes the **aggregate** capital stock.
- In equilibrium, they will be the same.
- But the household takes K as given when they make their decisions!
- So we can't allow them to internalise their choices' effect on K .

Concept (3) Recursive Competitive Equilibrium

- A recursive competitive equilibrium is a set of functions
 - Quantities $G(K)$, $g(k, K)$: the law of motion for aggregate capital and the household's policy function respectively.
 - Lifetime utility level $v(k, K)$.
 - Price $R(K)$, all such that

(1'') Value function $v(k, K)$ solves the household's recursive formulation and $g(k, K)$ is the associated policy function.

(2'') Prices are determined competitively

$$R(K) = \alpha K^{\alpha-1}$$

Concept (3) Recursive Competitive Equilibrium

(3'') Consistency is satisfied

$$G(K) = g(K, K) \quad \forall K$$

where notice that the requirement that the capital law of motion equal the household's policy function is an equilibrium condition.

- It's not something that we impose until **after** solving the household's problem with K given.
- This condition says that when the household is endowed with a level of capital equal to that of the aggregate, their behaviour is consistent with the aggregate law of motion.

Concept (3) Recursive Competitive Equilibrium

- Notice that the price $R(K)$ is a function as opposed to a price sequence like it was before.
- The same goes for the control variables: $c(k, K)$ and $k'(k, K)$.
- This is nice: we simplified the infinite-dimensional problem into something recursive and time-invariant that we can solve on a **computer!**

Roadmap

- 1 Introduction
- 2 Equilibrium Concepts in Dynamic Models
- 3 Extending to Stochastic Models**
- 4 Computing RCE via Value Function Iteration
- 5 Transition Dynamics Between RCE: Shooting Algorithm
- 6 Local Solutions: Perturbation Methods
- 7 Conclusion

Randomness

- With randomness, we need to think about states of the world in each period.
- Say we think now about a production function of the form $A_t K_t^\alpha$.
- Where A_t follows a Markov process.
- Means that current probabilities are determined by most recent realisations, (A_{t-1} is the state for the stochastic process).
- $A_t \sim Q(A_t | A_{t-1})$.
- Say that the process for A_t is discretised into $|\Omega|$ elements (countably finite), which are invariant across time.
- I.e. $A_t(\omega)$ where $\omega \in \{\omega_1, \omega_2, \dots, \omega_{|\Omega|}\}$.

Stochastic Recursive Competitive Equilibrium

- The notion of a recursive competitive equilibrium is extendible to a stochastic world, (assuming a Markov technology process).
- Again, just build-off the ideas of recursive household representations and consistency.
- Let's assume for now that the household can only save through capital, meaning that markets are **incomplete**.
- Everything is the same as before except output comes through $A_t K_t^\alpha$ where A_t follows a Markov process.

Stochastic Recursive Competitive Equilibrium

- Household's problem will now be given by

$$v(k, K, A) = \max_{\{c, k'\}} u(c) + \beta \mathbb{E}_{A'|A} [V(k', K', A')]$$

subject to

$$\begin{aligned}c + k' &= R(K, A)k + (1 - \delta)k \\ K' &= G(K, A)\end{aligned}$$

Stochastic Recursive Competitive Equilibrium

- A **stochastic** recursive competitive equilibrium is a set of functions
 - Quantities $G(K, A)$, $g(k, K, A)$: the law of motion for aggregate capital and the household's policy function respectively.
 - Lifetime utility level $v(k, K, A)$.
 - Price $R(K, A)$, all such that

(1'') Value function $v(k, K, A)$ solves the household's recursive formulation and $g(k, K, A)$ is the associated policy function.

(2'') Prices are determined competitively

$$R(K, A) = \alpha AK^{\alpha-1}$$

Stochastic Recursive Competitive Equilibrium

(3'') Consistency is satisfied

$$G(K, A) = g(K, K, A) \quad \forall K$$

- The state can indeed be changing between periods, (through fluctuating A).
- But again, the problem always looks the same in this recursive setup.

Roadmap

- 1 Introduction
- 2 Equilibrium Concepts in Dynamic Models
- 3 Extending to Stochastic Models
- 4 Computing RCE via Value Function Iteration**
- 5 Transition Dynamics Between RCE: Shooting Algorithm
- 6 Local Solutions: Perturbation Methods
- 7 Conclusion

Algorithm

- Hopefully you have a decent intuition for the procedure by now.
 - Say we are solving the deterministic model from the second section of these slides.
- (1) Guess the law of motion for aggregate capital $G(K)$.
 - (2) Find the return $R(K)$.
 - (3) Solve the household's problem in the standard way with VFI to get $g(k, K)$.
 - (4) Update your guess of $G(K)$, (given that the individual and aggregate functions are meant to be "close").

Algorithm

- This process is effectively mapping from a guess of $G(K)$ to an update of it.
- This functional is **not** necessarily a contraction.
- Iterations on the computer might not converge, if it does, the differences may be highly non-monotonic.
- **Update policy functions slowly!**

Algorithm

- You can use other methods rather than VFI here, (which can be faster).
- See the appendix for more details.

Roadmap

- 1 Introduction
- 2 Equilibrium Concepts in Dynamic Models
- 3 Extending to Stochastic Models
- 4 Computing RCE via Value Function Iteration
- 5 Transition Dynamics Between RCE: Shooting Algorithm**
- 6 Local Solutions: Perturbation Methods
- 7 Conclusion

Transition Dynamics

- I've been pushing recursive competitive equilibria on you all day.
- We were looking for functions of the state space that were invariant over time.
- What happens in the face of a government policy change though?
- Thing must be changing in the short-run as we transition to a **new RCE**.

Transition Dynamics

- Consider the social planner's formulation for the neoclassical growth model.
- I.e. let's step away from markets for a second just to keep things simple.
- The concepts are the same when you have a decentralised economy.

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = k_t^\alpha + (1 - \delta)k_t$$

for some initial k_0 .

Transition Dynamics

- Optimality condition for capital investment is

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + \alpha k_{t+1}^{\alpha-1}]$$

- Start at point k_0 and the model will eventually converge to a steady state given by

$$k^{ss} = \left\{ \left(\frac{1}{\beta} - (1 - \delta) \right) \frac{1}{\alpha} \right\}^{\frac{1}{\alpha-1}}$$

Transition Dynamics

- Say that the economy was in this steady state until time $t = 0$, (i.e. k_0 is equal to the initial steady state).
- Then the depreciation rate magically (unanticipated) increases to $\delta' > \delta$. Will stay with this depreciation rate forever more.
- The new steady state is given by

$$k^{ss'} = \left\{ \left(\frac{1}{\beta} - (1 - \delta') \right) \frac{1}{\alpha} \right\}^{\frac{1}{\alpha-1}} < k^{ss}$$

Transition Dynamics

- Ok...fine. How do we reach this new steady state though?
- We have initial and endpoint conditions.
- Also know the sufficient conditions for the optimum (Euler equation and resource constraint).
- Just need to map journey between the two steady states.

Shooting Algorithm

- (1) Guess the number of time periods it takes to transition to the new steady state. Call this number $S \in \mathbb{N}$.
- (2) **Guess** your initial value for consumption, c_0 .
- (3) This then implies your initial investment

$$\Rightarrow k_1 = (k^{ss})^\alpha + (1 - \delta)k^{ss} - c_0$$

- (4) Iterate on your Euler equation to get c_1

$$c_1 = u'^{-1} \{ u'(c_0) \beta^{-1} [1 - \delta + \alpha k_1^\alpha]^{-1} \}$$

which is just $c_1 = \beta c_0 [1 - \delta + \alpha k_1^{\alpha-1}]$ if log utility.

Shooting Algorithm

(5) Repeat this procedure until time S .

(6) Will give you a candidate transition path

$$\{c_0, c_1, \dots, c_S\} \text{ and } \{k_{SS}, k_1, k_2, \dots, k_S\}$$

check if $k_S = k^{SS'}$. Stop if sufficiently close.

(7) If not sufficiently close, update your guess of c_0

- If $k_S < k^{SS'}$ then lower c_0 .
- If $k_S > k^{SS'}$ then increase c_0 . Return to step (3).

Shooting Algorithm

- Pretty straightforward right?
- Things get more complicated if we don't have closed-form optimality conditions, (e.g. heterogeneous agents, discrete choices, more control variables).
- So we'll revisit this notion of transition dynamics when we get to heterogeneous agents models.

Roadmap

- 1 Introduction
- 2 Equilibrium Concepts in Dynamic Models
- 3 Extending to Stochastic Models
- 4 Computing RCE via Value Function Iteration
- 5 Transition Dynamics Between RCE: Shooting Algorithm
- 6 Local Solutions: Perturbation Methods**
- 7 Conclusion

Global v.s. Local Solutions

- The methods we've covered so far yield **global** solutions to problems.
- Gives you the solution to the problem over the entire domain, (or an approximation to it).
- Local solutions, in contrast, give you solutions in a small neighbourhood of some point.

Global v.s. Local Solutions

- When would we want to use a local solution?
- If we're thinking about small (temporary) deviations from a steady state.
- Again, emphasis on **small**.
- The approach is inaccurate for large deviations.

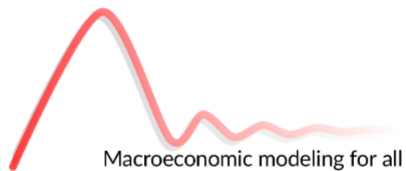
Dynare

- The good thing about local solutions is that the barrier to implementation is **incredibly low**.
- Some good citizens (predominantly based in France), developed a software called Dynare.
- Their current slogan says it all...

Dynare

← → ↻ 🏠 🔒 dynare.org

Dynare



[Download v4.5.7](#)

[Documentation](#)

Dynare

- This software is a toolbox for Matlab.
- Easy to install, easy to use.
- Solves and **estimates** dynamic rational expectations models with little-no programming experience.
- Can use maximum likelihood or Bayesian estimation.
- Very popular with people at central banks.
- Used **a lot** by researchers of the new Keynesian DSGE paradigm.

Dynare

- All you have to do is find the optimality conditions for your problem and type them into a simple script.
- Dynare uses local approximations to these optimality conditions to find your solution.
- You can either find the local approximation yourself, (I'll talk about this in a moment), or even get the software to do it for you.

Dynare

- The software's output depends on whether you're just solving or also estimating a model.
- Solving a model gives output of locally-approximated policy functions.
- Estimation also tells you the parameter values of the model that are **internally consistent** with the data you provide it with.

Dynare

- I'm all in favour of this software as it does make structural modelling super-accessible.
- To give you an idea: I had three undergraduates last year who studied new Keynesian models with Dynare.
- Really great stuff and a true service to the profession that these developers are contributing.

Controversial Statements

- There's always a *but* with these things though.
- I'm not going to teach you how to use it.
- I've made you aware of it and I'll explain the general idea behind perturbation methods in the slides to come.
- Coming from graduate school at Wisconsin, I can't in all good conscience teach you how to use Dynare here...

Controversial Statements

- My advisor once said:

These heavy computational guys like Victor Rios-Rull would never speak to you again if they found out that you used Dynare (Corbae, 2014).

- I think he might have also used the words “non-macho”, “non-kosher” and Dynare in the same sentence.
- If you want to be a serious quantitative economist, you need to code things up for yourself.
- Dynare’s like a **black box**. Fine for policymakers who want a quick quantitative estimate. Won’t get you very far (in terms of journal ranking) if you’re writing a paper you want to publish though.

Perturbation Methods

- Say that we're trying to solve a functional equation of the form

$$\mathcal{F}(x, x') = 0$$

Perturbation Methods

- The perturbation approach approximates using

$$x'(x, \tilde{b}) = \sum_{i=0}^n \tilde{b}_i (x - x_0)^i$$

where x_0 is a **particular point** and \tilde{b} is a vector of coefficients.

- The solution is analytic for a neighbourhood around x_0 .

Perturbations about Steady State

- The typical thing to do in economics is perturb the model about its **non-stochastic** steady state.
- Shut-down the shocks and find the point where all the variables are constant.
- Then look for a locally analytic solution.
- Although, in principle, you can approximate about any point.

Perturbations about Steady State

- Consider the stochastic growth model

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t c_t^{1-\sigma}$$

subject to

$$c_t + k_{t+1} = a_t k_t^\alpha + (1 - \delta)k_t$$

$$\log(a_t) = \rho \log(a_{t-1}) + \sigma \epsilon_t, \quad \epsilon_t \sim N(0, 1)$$

Perturbations about Steady State

- Euler equation for an arbitrary time period

$$c_t^{-\sigma} = \beta \mathbb{E}_t \left\{ c_{t+1}^{-\sigma} [\alpha a_{t+1} k_{t+1}^{\alpha-1} + (1 - \delta)] \right\}$$

- In the non-stochastic steady state, $\epsilon_t = 0 \forall t$. Gives

$$\begin{aligned} \log(a_t) &= \rho \log(a_{t-1}) \\ \Rightarrow a_t &= a^{ss} \\ &= 1 \end{aligned}$$

meaning that

$$\begin{aligned} 1 &= \beta \left\{ \alpha (k^{ss})^{\alpha-1} + (1 - \delta) \right\} \\ \Rightarrow k^{ss} &= \left\{ \left(\frac{1}{\beta} - (1 - \delta) \right) \frac{1}{\alpha} \right\}^{\frac{1}{\alpha-1}} \end{aligned}$$

just as before.

Perturbations about Steady State

- Also have the steady state resource constraint

$$c^{SS} = (k^{SS})^\alpha - \delta k^{SS}$$

Perturbations about Steady State

- Let's pick a perturbation parameter λ .
- In this example, λ is such that

$$\log(a_t) = \rho \log(a_{t-1}) + \lambda \sigma \epsilon_t$$

where $\lambda = 1$ is the stochastic case and $\lambda = 0$ is the deterministic steady state.

- We now search for decision rules of the form

$$c_t = c(k_t, a_t, \lambda)$$
$$k_{t+1} = k(k_t, a_t, \lambda)$$

Perturbations about Steady State

- We now seek a local approximation about the point $(k^{ss}, 1, 0)$ for (k_t, a_t, λ) .
- We want to approximate the policy functions $c = c(k, a)$ and $k' = k(k, a)$ locally.
- How? Taylor's theorem.

Taylor's Theorem

- The policy function expansions are of the form

$$\begin{aligned}c_t &= c(k^{ss}, a_t, 1) \\ &= c(k^{ss}, 1, 0) \\ &\quad + c_k(k^{ss}, 1, 0)(k_t - k^{ss}) + c_a(k^{ss}, 1, 0)(a_t - 1) + c_\lambda(k^{ss}, 1, 0)\lambda \\ &\quad + \frac{1}{2}c_{kk}(k^{ss}, 1, 0)(k_t - k^{ss})^2 + \frac{1}{2}c_{ka}(k^{ss}, 1, 0)(a_t - 1)(k_t - k^{ss}) + \dots\end{aligned}$$

look familiar?

- Same idea for the k policy function.

Taylor's Theorem

- Recall we had two equilibrium conditions, which we'll now denote by

$$\vec{F}(k_t, a_t, \lambda) = \mathbb{E}_t \left[\begin{array}{l} c(k_t, a_t, \lambda)^{-\sigma} - \beta \left\{ (c(k(k_t, a_t, \lambda), a_{t+1}, \lambda))^{-\sigma} [\alpha a_{t+1} k(k_t, a_t, \lambda)^{\alpha-1} + (1-\delta)] \right\} \\ c(k_t, a_t, \lambda) + k(k_t, a_t, \lambda) - (1-\delta)k_t - a_t k_t^\alpha \end{array} \right]$$

where $\vec{F}(k_t, a_t, \lambda) = \vec{0}$ from our FOCs.

- We can also denote this as

$$\vec{F}(k_t, a_t, \lambda) = \vec{\mathcal{F}}(c_t, c_{t+1}, k_t, k_{t+1}, a_t, \lambda)$$

where I write it in this way to make explicit that the dependence through the states come through policy functions for c_t , c_{t+1} and k_{t+1} .

- Denote the derivative of the j^{th} entry of $\vec{\mathcal{F}}$ by $\vec{\mathcal{F}}_j$

Zeroth-Order Expansion

- See that

$$\vec{F}(k^{ss}, 1, 0) = \vec{0}$$

$$\Rightarrow k^{ss} = \left\{ \left(\frac{1}{\beta} - (1 - \delta) \right) \frac{1}{\alpha} \right\}^{\frac{1}{\alpha-1}}$$

$$\Rightarrow c^{ss} = (k^{ss})^\alpha - \delta k^{ss}$$

i.e. just our steady state conditions.

First-Order Expansion

- Take first order derivatives and see that

$$\vec{F}_k(k^{ss}, 1, 0) = 0$$

$$\vec{F}_a(k^{ss}, 1, 0) = 0$$

$$\vec{F}_\lambda(k^{ss}, 1, 0) = 0$$

why the zeros?

First-Order Expansion

- Zeros follow from the fact that $\vec{F}(k_t, a_t, \lambda) = \vec{0}$ always.
- See then that

$$\vec{F}_k(k, 1, 0) = \vec{\mathcal{F}}_1 c_k + \vec{\mathcal{F}}_2 c_k k_k + \vec{\mathcal{F}}_3 + \vec{\mathcal{F}}_4 k_k = 0$$

$$\vec{F}_a(k, 1, 0) = \vec{\mathcal{F}}_1 c_a + \vec{\mathcal{F}}_2 \left[c_k k_a + c_a \frac{\partial a_{t+1}}{\partial a_t} \right] + \vec{\mathcal{F}}_4 k_a + \vec{\mathcal{F}}_5 = 0$$

where c and k denote the policy functions for the controls.

First-Order Expansion

- Where

$$\vec{F}_k(k, 1, 0) = \vec{\mathcal{F}}_1 c_k + \vec{\mathcal{F}}_2 c_k k_k + \vec{\mathcal{F}}_3 + \vec{\mathcal{F}}_4 k_k = 0$$

$$\vec{F}_a(k, 1, 0) = \vec{\mathcal{F}}_1 c_a + \vec{\mathcal{F}}_2 \left[c_k k_a + c_a \frac{\partial a_{t+1}}{\partial a_t} \right] + \vec{\mathcal{F}}_4 k_a + \vec{\mathcal{F}}_5 = 0$$

is a quadratic system of 4 unknowns (c_k, c_a, k_k, k_a) with 4 equations (given that we have both the Euler equation and resource constraint).

- Quadratic since we have these coefficients in cross products and the like.

Second-Order Expansion

- For this we then take the second derivatives around $(k, 1, 0)$.

$$F_{kk}(k, 1, 0) = 0$$

$$F_{ka}(k, 1, 0) = 0$$

$$F_{k\lambda}(k, 1, 0) = 0$$

$$F_{aa}(k, 1, 0) = 0$$

$$F_{a\lambda}(k, 1, 0) = 0$$

$$F_{\lambda\lambda}(k, 1, 0) = 0$$

How Many Orders?

- There are some things to note.
- First order approximations miss some things in relation to uncertainty.

How Many Orders?

- Fernandez-Villaverde et al. (2016) point out the following drawbacks of first order approximations
 - Hard to infer the welfare effects of uncertainty,
 - Solution can't generate risk premia for assets,
 - Can't study the consequences of a change in volatility.
- Ok...so go higher...more burdensome computationally though.
- Solving for more and more unknowns.

Roadmap

- 1 Introduction
- 2 Equilibrium Concepts in Dynamic Models
- 3 Extending to Stochastic Models
- 4 Computing RCE via Value Function Iteration
- 5 Transition Dynamics Between RCE: Shooting Algorithm
- 6 Local Solutions: Perturbation Methods
- 7 Conclusion**

Summary

- Basically covered two solution techniques.
- Recursive competitive equilibrium.
- Local approximations.
- Both have advantages and drawbacks....the appropriate method really depends on the application you have in mind.