Lecture 3: Real Business Cycle Model

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Roadmap

DSGE Models

- DSGE: Dynamic Stochastic General Equilibrium.
- Optimising representative agents and rational expectations.
- Controversial.
- Policy implications? Predictors of crises?
- Lots of work since the crisis on incorporating financial frictions.
- Last few years: relax the representative agent assumption; how do changes at the cross section affect aggregates?
- A quirky yet scathing review can be found in Quiggin (2010), "Zombie Economics: How Dead Ideas Still Walk Among Us".

Background

- RBC models are the original DSGEs.
- Combine microfoundations, dynamics and stochastic shocks to provide a theory of business cycle fluctuations.
- Consistent with the basic neoclassical growth model in the long-run.
- Exogeneous shocks (good assumption?) drive short-run fluctuations.
- Brock, W., & Mirman, L. (1972): "Optimal Economic Growth and Uncertainty: The Discounted Case", *Journal of Economic Theory*, 4(3), 479– 513.
- Kydland, F & Prescott, E. (1982) "Time to Build and Aggregate Fluctuations", *Econometrica*, 50: 1345-1370.

Preview of the Punch-Line

- Business cycles are a natural part of life.
- Business cycles are efficient: can eventuate even without any market failures.
- Decentralised market equilibrium achieves the efficient allocation of resources.
- Business cycles are endogenous fluctuations, which are induced by shocks coming from external forces.
- Role of government should not be on smoothing business cycles: focus instead on structural reforms.

Where are we Going with This?

- We want to study money with mathematical rigour in general equilibrium.
- Build-up to that one step at a time though.
- Before we start talking about nominal variables, this RBC model is entirely real.
- Look at the entirely real model this lecture, then add-in money and see what happens next lecture.

Roadmap

Fundamentals

- Representative agents: firms and households.
- Infinite horizon and discrete time $t \in \{0, 1, 2, 3, ...\}$.
- Perfectly competitive markets.
- General equilibrium.
- Real model: no role for money in this lecture.
- Prices are denoted in terms of real variables (e.g. goods or labour).

Households Setup

- Supply labour to firms and own the capital stock, (rented out to firms).
- Objective is to maximise the expected present value of their lifetime utility subject to period-by-period budget constraints.
- Discounting over time: constant discount factor $0 < \beta < 1$, (money tomorrow is worth less than money today due to opportunity cost).
- Time separable utility.
- Household owns the firm and receives its profits as income d_t.

Households' Problem

- Problem:

$$\max_{\{c_t, n_t, i_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\psi}}{1+\psi} \right]$$

subject to their budget constraints and law of motion for capital

$$\begin{aligned} c_t + i_t &\leq w_t n_t + r_t k_t + d_t \\ k_{t+1} &= i_t + (1-\delta) k_t \\ k_{t+1} &\geq 0 \ \forall t \\ k_0 & \text{ given} \end{aligned}$$

- How does this differ from the infinite horizon optimisation problem from last class?

Firms' Problem

- Static problem since they rent factor inputs:

$$\max_{\{k_t, n_t\}} d_t = y_t - w_t n_t - r_t k_t$$

where $y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$ and

$$\log(a_t) = \rho \log(a_{t-1}) + \epsilon_t, \ \ \epsilon_t \sim N(0, 1)$$

where $0 < \rho < 1$.

- Zero profits $d_t = 0$.

Roadmap

Household Optimality: Lagrangian

- Lagrangian (substitute out investment for capital law of motion)

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\psi}}{1+\psi} \right] +$$

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \lambda_t \left[w_t n_t + (1-\delta+r_t)k_t + d_t - c_t - k_{t+1} \right]$$

Household Optimality: First Order Conditions

- Notice that $\mathbb{E}_t[x_t] = x_t$.
- FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - \lambda_t = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Rightarrow -\beta^t n_t^{\psi} + \lambda_t w_t = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Rightarrow -\lambda_t + \mathbb{E}_t[\lambda_{t+1}(1 - \delta + r_{t+1})] = 0 \tag{3}$$

Firm Optimality: First Order Conditions

- FOCs:

$$\frac{\partial d_t}{\partial k_t} = 0 \Rightarrow \alpha a_t k_t^{\alpha-1} n_t^{1-\alpha} - r_t = 0 \tag{4}$$

$$\frac{\partial d_t}{\partial n_t} = 0 \Rightarrow (1-\alpha) a_t k_t^{\alpha} n_t^{-\alpha} - w_t = 0 \tag{5}$$

Equilibrium Definition

- The competitive equilibrium of the RBC model is defined as a sequence of prices $\{w_t, r_t\}_{t=0}^{\infty}$ and allocations $\{c_t, k_{t+1}, n_t\}$ with the state vector $\{k_t, a_t\}$ taken as given by the agents in the model. Optimality conditions (1) – (5) above hold and all markets clear.

Canonical Representation

- Consolidate the household's FOCs to get labour supply and consumption Euler equation. Resource constraint from household's budget constraint.

- (1) and (2) give labour supply

$$c_t^{\sigma} n_t^{\psi} = w_t$$

- (3) and (1) give the consumption Euler equation

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} (1 - \delta + r_{t+1}) \right]$$

- Household budget constraint, $d_t = 0$ and (4) – (5) give the resource constraint

$$c_t + i_t = w_t n_t + r_t k_t = y_t$$

- Look familiar?....

Social Planner's Problem and Efficiency

- When studying market economies, we want a benchmark, against which we can compare the allocation of resources.
- Social planner's problem: solves for the optimal allocation subject only to a physical resource constraint.
- The solution to the social planner's problem is efficient.
- Solve the social planner's problem and compare the optimality conditions with the market case: how well does the market economy do? Does it come close to the optimal allocation?

Social Planner's Problem

- Social planner's problem:

$$\max_{\{c_t, n_t, i_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\psi}}{1+\psi} \right]$$

  subject to their resource constraints and law of motion for capital

$$\begin{aligned} c_t + i_t &= y_t \\ k_{t+1} &= i_t + (1-\delta)k_t \\ k_{t+1} &\geq 0 \ \forall t \\ k_0 &\text{ given} \end{aligned}$$

- The solution to this program is Pareto optimal.
- Exercise: show that this program yields the same solution as the RBC market economy above.

Roadmap

What's Going on in this Model?

- The solution is Pareto optimal, yet random shocks are still present.
- The productivity process a_t drives everything in this model!
- It's exogenous: philosophical implication?
- Shocks to productivity drive endogenous responses in other variables.
- We'll study local (small) deviations from a steady state as the solution.

Roadmap

What's a Steady State?

- The steady state of a model is defined as a situation in which variables are unchanging over time.
- In this model, this means $a_t = a_{t-1} = 1$ (as $\rho < 1$).
- As a consequence, $c_t = c_{t-1} = \bar{c}$, $k_t = k_{t-1} = \bar{k}$ etc for endogenous variables.
- The steady state is what prevails when we shut-down all the randomness in the model.

What Does the Steady State Look Like?

- Steady state labour supply

$$\bar{c}^{\sigma}\,\bar{n}^{\psi} = \bar{w} \tag{6}$$

- Steady state Euler equation

$$1 = \beta(1 - \delta + \bar{r}) \tag{7}$$

- Steady state resource constraint

$$\bar{c} + \bar{i} = \bar{y} \tag{8}$$

What Does the Steady State Look Like?

- From (4) and (5), the steady state factor prices are

$$\bar{r} = \alpha \bar{k}^{\alpha-1} \bar{n}^{1-\alpha} \tag{9}$$
$$\bar{w} = (1-\alpha) \bar{k}^{\alpha} \bar{n}^{-\alpha} \tag{10}$$

- From the capital law of motion, steady state investment is

$$\bar{i} = \delta \bar{k} \tag{11}$$

- From the production function, steady state output it

$$\bar{y} = \bar{k}^{\alpha} \bar{n}^{1-\alpha} \tag{12}$$

- From the technological process

$$\bar{a} = 1 \tag{13}$$

What Does the Steady State Look Like?

- Equations (6) – (13) define the steady state.
- Eight equations in eight unknowns $\{\bar{c}, \bar{n}, \bar{w}, \bar{r}, \bar{i}, \bar{y}, \bar{k}, \bar{a}\}$.
- Now we approximate small deviations about the steady state when shocks are present using log-linearisation.

Roadmap

Log-Linearisation

- Linearised labour supply

$$
\begin{aligned}
[\bar{c}e^{\hat{c}_t}]^{\sigma}[\bar{n}e^{\hat{n}_t}]^{\psi} &= \bar{w}e^{\hat{w}_t} \qquad (14)\\
\Rightarrow \bar{c}^{\sigma}\,\bar{n}^{\psi}[e^{\sigma\hat{c}_t+\psi\hat{n}_t}] &= \bar{w}e^{\hat{w}_t}\\
\Rightarrow e^{\sigma\hat{c}_t+\psi\hat{n}_t} &= e^{\hat{w}_t}\\
\Rightarrow 1+\sigma\hat{c}_t+\psi\hat{n}_t &\approx 1+\hat{w}_t\\
\Rightarrow \sigma\hat{c}_t+\psi\hat{n}_t &\approx \hat{w}_t
\end{aligned}
$$

where the first line comes from the definition of $\hat{x}_t$ [see lecture 1] and the penultimate line comes from a Taylor expansion of first order of line 3.

Log-Linearisation

- Linearised Euler equation

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] + \frac{\bar{r}}{\sigma(1-\delta)}\mathbb{E}_t[\hat{r}_{t+1}] \tag{15}$$

- Linearised resource constraint

$$\hat{y}_t = \frac{\bar{c}}{\bar{y}}\hat{c}_t + \frac{\bar{i}}{\bar{y}}\hat{i}_t \tag{16}$$

- Linearised factor prices

$$\hat{r}_t = \hat{a}_t + (\alpha - 1)\hat{k}_t + (1-\alpha)\hat{n}_t \tag{17}$$

$$\hat{w}_t = \hat{a}_t + (\alpha)\hat{k}_t + (-\alpha)\hat{n}_t \tag{18}$$

Log-Linearisation

- Linearised capital law of motion

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta\hat{i}_t \tag{19}$$

- Linearised production function

$$\hat{y}_t = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{n}_t \tag{20}$$

- Linearised technology process

$$\hat{a}_t = \rho\hat{a}_{t-1} + \epsilon_t \tag{21}$$

Log-Linearisation

- Exercise: derive (15) – (21) yourself.
- Again we have eight equations in eight unknowns $\{\hat{c}_t, \hat{n}_t, \hat{w}_t, \hat{r}_t, \hat{i}_t, \hat{y}_t, \hat{k}_t, \hat{a}_t\}$.

Roadmap

Back to the Big Picture

- It's possible to show that the control variables can be written as a linear function of (a, k) in this linearised system.
- You don't need to show this yet: I'll teach you about this in the solving DSGEs lecture on analytical solutions.
- Just note for now that you can write

$$\begin{aligned}
\hat{n}_t &= \eta_{n,a}\hat{a}_t + \eta_{n,k}\hat{k}_t \\
\hat{k}_{t+1} &= \eta_{k',a}\hat{a}_t + \eta_{k',k}\hat{k}_t \\
\hat{c}_t &= \eta_{c,a}\hat{a}_t + \eta_{c,k}\hat{k}_t.
\end{aligned}$$

Back to the Big Picture

- This system of variables respond endogenously to productivity shocks.
- E.g. say we start in steady state at $t = 0$ and then $\hat{a}_1 = \epsilon_1$ then no further shocks (called an impulse response).
- We can trace-out the time paths for the endogenous variables:

$$\hat{n}_1 = \eta_{n,a}\epsilon_1$$
$$\hat{k}_2 = \eta_{k',a}\epsilon_1$$
$$\hat{c}_1 = \eta_{c,a}\epsilon_1$$

$$\hat{n}_2 = \eta_{n,a}[\rho\epsilon_1] + \eta_{n,k}[\eta_{k',a}\epsilon_1]$$
$$\hat{k}_3 = \eta_{k,a}[\rho\epsilon_1] + \eta_{k',k}[\eta_{k',a}\epsilon_1]$$
$$\hat{c}_2 = \eta_{c,a}[\rho\epsilon_1] + \eta_{c,k}[\eta_{k',a}\epsilon_1]$$

$$\ldots\ldots$$

Back to the Big Picture

- Under certain stability conditions for the parameters (to be discussed later), we'll eventually converge back to steady state.