Lecture III
Appendix: Discrete Choices

Adam Hal Spencer

The University of Nottingham

Applied Computational Economics 2020

Roadmap

Motivation

- So far we've looked at models where agents optimally choose control variables on a continuum (e.g. consumption).
- It's often also natural to think about what are called discrete choices.
- In the household setting, the most obvious thing to think about is consumer default.
- Should I pay-off my loan or just file for bankruptcy?

Motivation

- The default example has a binary choice variable that takes the values of default or don't default.
- More generally, (especially in the firm case), we can think of multi-valued discrete choices.
- E.g. in my old JMP, I had firms choosing whether to exit, be a domestic firm, an exporter of a multinational, (these are often referred to as Melitz models).

Consumer Default Model

- Let's consider a basic model where households have the option to strategically default.
- Recall in Huggett (1993) there was a collateral constraint and a commitment technology, which ensured people would always repay their debt.
- You can think of their debt as being "collateralised" in some sense.
- An alternative is to think about unsecured debt.
- We'll add this twist now to the model we considered a few sections ago.

Consumer Default Model Setup

- Unit mass of households.
- Standard preferences: $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$.
- Still face earnings shocks: $s_t \in \{e, u\}$, which denotes employment and unemployment respectively.
- When employed, household gets $y(s_t = e) = 1$, in contrast with $y(s_t = u) = b < 1$; follows a Markov process.

Consumer Default Model Setup

- Households can save through one-period bonds, which are now unsecured.
- What is the price at which households can save/borrow?
- Denote the asset holdings of a household by a_{t+1}.
- The household's discrete choice for the period, we'll denote by $x_t(a_t, s_t) \in \{D, N\}$ where D stands for default and N stands for no default.
- Notice that the default choice is state-contingent.

Consumer Default Model Setup

- If $a_{t+1} \geqslant 0$ (saving), the household receives the riskless rate with bond price $q_t = \frac{1}{1+r}$.
- If $a_{t+1} < 0$ (borrowing), the household pays a price that accounts for the possibility of default.
- Let's stop for a moment and think about this in a world with a single borrower...
- Assume that the lender is competitive: meaning that the bond price they offer will lead to them breaking-even in expectation.

Consumer Default Model Setup

- The bond price for this single borrower comes from

$$q_t = \frac{1}{1+r}\mathbb{E}_t\left\{\mathbb{1}_{[x_{t+1}(a_{t+1},s_{t+1})=D]}(0) + \mathbb{1}_{[x_{t+1}(a_{t+1},s_{t+1})=N]}(1)\right\}$$

- This just says that the price of a bond equals the expected discounted value of what the lender will receive next period.

- Here we've assumed that there is no recovery/liquidation value accumulated by the lender in the event of default.

- In principle, this would just replace the (0) payoff in the event of default.

Consumer Default Model Setup

- See then that

$$
\begin{aligned}
q_t &= \frac{1}{1+r}\mathbb{E}_t\left\{\mathbb{1}_{[x_{t+1}(a_{t+1},s_{t+1})=D]}(0) + \mathbb{1}_{[x_{t+1}(a_{t+1},s_{t+1})=N]}(1)\right\} \\
&= \frac{1}{1+r}\mathbb{E}_t\left[\mathbb{1}_{[x_{t+1}(a_{t+1},s_{t+1})=N]}\right] \\
&= \frac{1}{1+r}\Pr(x_{t+1}(a_{t+1},s_{t+1}) = N)
\end{aligned}
$$

  that is — the price equals the discounted probability of the household repaying their debt.

- If this probability equals one, the price is the same as that for a riskless bond.

- Lower probability of repayment reduces the bond price, (which recall increases the interest rate).

Consumer Default Model Setup

- A little more complicated when we have multiple borrowers.
- Two approaches are to think about either pooling or separating contracts.
- Separating contracts offer different prices/interest rates to borrowers based on their characteristics.
- Pooling contracts treat everyone the same.
- We'll focus on pooling contracts, (more to come in a few slides' time).

Consumer Default Model Setup

- In the U.S., the most common type of bankruptcy is referred to as Chapter 7 bankruptcy.
- The "fresh start". Your debts are written-off and you head back to go.
- You have a flag attached to your credit record for 10 years though.

Consumer Default Model Setup

- Model this in a stylised way.
- In the event of default, your debt a_t is removed.
- A flag is placed on your credit history. Denote this $h_{t+1} = 1$.
- If $h_{t+1} = 1$, then you can't borrow.
- Can only save $a_{t+1} \geqslant 0$.

Consumer Default Model Setup

- What's the complicating factor about having this "10 year business' with the flags in our modelling setup?
- Rather than introducing a state for the number of years a household has had the flag for, instead say there is some probability their flag will be removed in a given period of $1 - \rho \in [0, 1]$.
- Calibrate ρ such that $\frac{1}{1-\rho} = 10$ is the average duration of a flag.

Consumer Default Model Setup

- Timing:
  - **(1)** Enter period t with a_t and h_t.
  - **(2)** Realise the earnings shock, (i.e. $s_t \in \{e, u\}$).
  - **(3a)** If $h_t = 0$ and $a_t < 0$ make default decision $x_t(a_t, s_t) \in \{D, N\}$.
    - If $x_t(a_t, s_t) = 1$ then $a_{t+1} = 0$ and $h_{t+1} = 1$.
    - If $x_t(a_t, s_t) = 0$ then choose a_{t+1} (unrestricted) and $h_{t+1} = 0$.
  - **(3b)** If $h_t = 1$ then make savings decision $a_{t+1} \geqslant 0$ and $h_{t+1} = 0$ with probability $1 - \rho$.

Consumer Default Model Equilibrium

- Discrete choices like these can be integrated beautifully into the dynamic programming setup.
- Denote a household's value function by $V(a, s, h)$.

Consumer Default Model Equilibrium

- Household's problem if $h = 0$ given by

$$V(a, s, 0) = \max_{\{N,D\}} \left\{V^{x=N}(a, s, 0), V^{x=D}(a, s, 0)\right\}$$

where

$$V^{x=N}(a, s, 0) = \max_{\{a'\}} \; u\left(y(s) + a - qa'\right) + \beta\mathbb{E}[V(a', s', 0)]$$

$$V^{x=D}(a, s, 0) = u(y(s)) + \beta\mathbb{E}[V(0, s', 1)]$$

Consumer Default Model Equilibrium

- Household's problem if $h = 1$ given by

$$V(a,s,1) = \max_{a' \geqslant 0} \; u\left(y(s) + a - \frac{a'}{1+r}\right) + \beta \mathbb{E}\left[\rho V(a',s',1) + (1-\rho) V(a',s',0)\right]$$

- The solution to the household's problem will then be decision rules $x(a,s,0)$ and $a' = g(a,s,h)$.
- Where notice that you obviously can't default if you already have a flag.

Consumer Default Model Equilibrium

- We now have a cross-sectional law of motion that depends on the flag state.
- See that for $h_t = 0$

$$\mu'(a', s', 0) = \sum_s \left\{ \int_a \mathbb{1}_{a'=g(a,s,0)} [1 - x(a, s, 0)] \mu(da, s, 0) \right\} \pi(s, s') + (1-\rho) \sum_s \left\{ \int_a \mathbb{1}_{a'=g(a,s,1)} \mu(da, s, 1) \right\} \pi(s, s')$$

Consumer Default Model Equilibrium

- Then for $h_t = 1$

$$\mu'(a', s', 1) = \sum_s \left\{ \int_a \mathbb{1}_{a'=g(a,s,0)}[x(a,s,0)]\mu(da,s,0) \right\} \pi(s,s') + \rho \sum_s \left\{ \int_a \mathbb{1}_{a'=g(a,s,1)}\mu(da,s,1) \right\} \pi(s,s')$$

Consumer Default Model Equilibrium

- What does the lender's problem look like?
- Recall that we'll focus on pooling contracts, (everyone is offered the same borrowing price).
- Let's assume again that the lenders breakeven in expectation.

Consumer Default Model Equilibrium

- Assume that there are L loans made out today.
- Assume also that D' is the amount of those loans that will default tomorrow.
- The profit made on these L loans is then given by

$$\frac{L - D'}{1 + r} - qL$$

where q is the pooling price.

Consumer Default Model Equilibrium

- Here is a very subtle point about the equilibrium behaviour of the lender.
- The lender, (just like the borrower), does not know what earnings shock the borrower will receive in the future.
- They are, however, aware of their behaviour contingent upon each potential shock.
- The lender therefore uses the decision rules of the household to price the debt.

Consumer Default Model Equilibrium

- Recall that the lenders breakeven in expectation. This means then that

$$\frac{L - D'}{1 + r} - qL = 0$$
$$\Rightarrow q = \frac{1 - \Delta'}{1 + r}$$

where $\Delta' = \frac{D'}{L}$ denotes the loan loss rate.

Consumer Default Model Equilibrium

- We can use our trusty cross-sectional distribution to find these aggregate variables as

$$D' = \sum_{s'} \left[\sum_{s} \int_{a} \mathbb{1}_{g(a,s,0)<0} x(g(a,s,0), s', 0) \pi(s'|s) \mu(da, s, 0) \right]$$

$$L = \sum_{s} \int_{a} \mathbb{1}_{g(a,s,0)<0} g(a,s,0) \mu(da, s, 0)$$

Consumer Default Model Equilibrium

- We're now in a position to define a recursive competitive equilibrium in the context of heterogeneous agents!

Consumer Default Model Equilibrium

- A recursive competitive equilibrium is a list of value functions, decision rules, a bond price and a cross-sectional distribution such that
    - Taking price q as given, $\{V(a, s, h; q), x(a, s, h; q)\}$ satisfies the household optimisation problem.
    - The bond price is such that the lender makes zero profits.
    - There is a fixed point of the law of motion for the cross-sectional distribution.

Consumer Default Model Equilibrium

- Computational algorithm.

(1) Make an initial guess for the bond price q_0, (e.g. the riskless price).

(2) Solve the optimisation problem for the household and compute the stationary cross-sectional measure.

(3) Check to see if the lenders are breaking even. I.e.

$$\left| q_0 - \frac{1 - \Delta'(q_0)}{1 + r} \right| < \epsilon$$

for some small ϵ. If yes, stop. If no, update the guess and return to step (2).

Consumer Default Model Equilibrium

- How should the guess be updated?
- You can try bisection. But sometimes there are issues with convergence.
- These models are a lot more complicated theoretically than the Huggett (1993) model without default.
- The size of the breakeven criterion under step (3) tends to behave in a very nonlinear way.

Consumer Default Model Equilibrium

- An alternative approach could be

$$q_1 = q_0 - \eta \left(q_0 - \frac{1 - \Delta'(q_0)}{1 + r} \right)$$

for some η sufficiently small that the algorithm converges.

Discrete Choices: Summary

- We can make the agent's decision problem much richer with these discrete choices.
- Issue though is that it introduces discontinuities.
- Means we're stuck with good old-fashioned value function iteration to solve the problem.
- Faster techniques like policy function iteration don't like these discontinuities.