Lecture IV
Solving Heterogeneous Agent General Equilibrium Models with Aggregate Uncertainty

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Roadmap

Motivation

- Yesterday we talked about Huggett (1993).
- Solved for the stationary distribution.
- Since there were no aggregate shocks, this thing wasn't changing at the aggregate level.
- But still movement at the idiosyncratic level, (like a Markov process).
- Beautiful stuff.

Motivation

- What's the problem with that though?
- Business cycles, anyone?
- We want fluctuations. The real macroeconomy bounces around a whole lot.

Motivation

- Recall that in the last lecture, we thought about idiosyncratic uncertainty in an endowment economy (as in Huggett (1993)).
- The way we'll proceed today is as follows
  (1) Idiosyncratic uncertainty in a production economy, (Aiyagari, 1994).
  (2) Using a model to simulate a panel dataset.
  (3) Aggregate and idiosyncratic uncertainty in a production economy, (Krusell and Smith, 1998).
- We'll delve into simulation as it turns-out to be important for solving models with aggregate uncertainty.
- Also relevant for estimation: I'll talk about that a bit as well.

Roadmap

Environment

- Take a unit measure of agents with preferences $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$.
- Agents supply their labour, (inelastically since there's no disutility) for a wage, w_t.
- Agents hold capital $k_t \in \mathscr{K} = [\underline{k}, \infty)$, which yields a return of r_t.
- Multiplicative employment shock $e_t \in \mathscr{E}$ such that $w_t \bar{e} e_t$ is the agent's labour income.
- Interpret parameter $\bar{e}$ as the fixed time endowment for labour supply.
- Assume that e_t follows a Markov process.

Environment

- There is a representative firm that has CRS technology $Y_t = F(K_t, L_t)$ where K_t is capital hired and L_t is labour hired.
- Capital depreciates at rate $\delta \in [0, 1]$.

Equilibrium

- The household's recursive formulation is given as

$$V(k,e) = \max_{c,k'} \; u(c) + \beta\mathbb{E}[V(k',e')]$$

subject to

$$\begin{aligned} k' + c &= w\bar{e}e + (1-\delta+r)k \\ c &\geqslant 0 \\ k &\geqslant 0. \end{aligned}$$

- The agents take w and r as given.

Equilibrium

- The firm's optimisation problem is given by

$$\max_{K,L} Y - wL - rK$$

where r and w are taken as given.

Equilibrium

- A recursive competitive equilibrium is defined as prices w, r, optimal decision rules, a value function $v(k, e)$, a cross-sectional distribution of agents $\mu(k, e)$ and the aggregate capital stock K and labour L such that

  (a) Given prices w and r, value function $v(k, e)$ is a solution to the household's optimisation problem and it has associated asset and consumption decision rules,

  (b) Prices r and w satisfy

$$
\begin{aligned}
r &= F_K(K, L) \\
w &= F_L(K, L)
\end{aligned}
$$

Equilibrium

(c) The stationary distribution $\mu(k, e)$ comes from the agent's decision rules and Markov transition probability for e.

(d) The aggregate capital and labour stock are consistent with the stationary distribution

$$K = \int_k k\mu(dk, e)$$
$$L = \bar{e}\int_e e\mu(k, de)$$

Solution Algorithm

- Recall that the labour supply of households is inelastic.
- The two equilibrium objects we need to compute are the prices w and r.
- But notice that the only real thing these depend on are K.
- So finding K really is all we need to do.

Solution Algorithm

- See the similarity to Huggett (1993), here?
- We can just use the excess demand equation for capital to find the equilibrium r.
- Once this is in equilibrium, we've also found w due to the nature of the problem.
- Huggett (1993) and Aiyagari (1994) are basically the same.

Roadmap

Motivation

- There are tons of reasons why you'd want to use your model to build a panel dataset.
- I'll quickly touch on two motivations here and then discuss the implementation.

Motivation 1: Talking to Empirical Researchers

- Most economists use reduced-form methodologies.
- We can still speak to them with these models.
- Once solved, we have a population of agents.
- Draw a sequence of earnings shocks, let the world play out and you can then run regressions.

Motivation 1: Talking to Empirical Researchers

- These models are laboratories.
- The problems associated with reduced-form are circumvented here because we control the environment!
- No issues with endogeneity or any of those terrifying things.

Motivation 1: Talking to Empirical Researchers

- Famous example in corporate finance: the investment-cash flow sensitivity.
- Fazzari, Hubbard and Petersen (1988) ran regressions of firm-level investment against cash flows (and Tobin's Q).
- Found that firms, which they postulate are more financially constrained based on observable characteristics, have a larger regression coefficient on cash flow.
- Claim was that the regression coefficient on cash flow is a proxy for financial constraints.

Motivation 1: Talking to Empirical Researchers

- These claims almost started the third world war amongst empirical corporate finance researchers.
- Tons of studies found evidence against and in favour of Fazzari et al. (1998)'s claims.
- Then a modelling guy came along.
- Gomes (2001, AER) found that the patterns of Fazzari et al. (1998) could emerge even in a model without financial frictions.
- He could make these statements because he had a controlled laboratory.

Motivation 2: Estimation

- Speaking to empiricists is one motivation for building your own dataset with a model.
- A more immediate motivation is that you can estimate parameters of your model using simulations.
- Simulated method of moments (SMM).

Motivation 2: Estimation

- The name sounds a lot like the generalised method of moments (GMM).
- SMM is the same idea except we use the dataset we built using our model.
- The idea is to choose parameters in your model to match the data moments.

Motivation 2: Estimation

- Say we're trying to estimate a parameter b.
- Denote M_d as a vector of data moments and $M_m(b)$ as a vector of simulated model moments.
- The estimator

$$\hat{b} = \arg\min[M_d - M_m(b)]' W [M_d - M_m(b)]$$

  is a consistent and asymptotically normal estimator of b under certain conditions.
- Note that W is a weighting matrix.

Motivation 2: Estimation

- Calibration is just a special case of this where $W = I$.
- Need the optimal weighting matrix to get standard errors on the estimated parameters.
- Two stage procedure that requires a simulated dataset.
- Effectively, you're changing your parameters and re-solving the model over and over again until your model and data moments are close.

Pseudo-Random Numbers

- In Matlab, getting a "random" draw of numbers is as simple as

```
X = rand(100);
```

  which generates a 100×100 matrix of numbers drawn from a uniform distribution in $(0, 1)$.

- Notice though that these numbers aren't truly random in the sense that they're selected using a deterministic algorithm.

Pseudo-Random Numbers

- Without getting to philosophical here, what is the implication of the *pseudo* part for our simulation procedures?
- If you're doing multiple draws for the purpose of estimation, then keep the seed of the draws the same when generating draws of agent-level shocks!
- The seed is a number that initialises the generator.
- If you re-initialise, this ensures that the same numbers will come up again.
- I.e. if you're changing parameters, simulating and then generating moments, you want to keep the seed the same each time to be sure that the moment changes are due to the parameter change rather than the alternative draws!

Pseudo-Random Numbers

- For example in Matlab

```
rng(1); % this sets the seed
X = rand(100);
rng(1);
Z = rand(100);
```

then X and Z are the same matrix. In contrast

```
rng(1);
X = rand(100);
Z = rand(100);
```

will generate X and Z as different matrices since the seed was not re-initialised.

Implementation

- The nice thing about using a Markov process for your exogenous shocks is that it makes simulating draws simple.
- Say that we have two agents and five time periods. They have two potential income shocks $e_t \in \{e^1, e^2\}$ for $t = 0, 1, 2, 3, 4$.
- Their initial draws come from a vector $\bar{Q}(e_0)$ and subsequent draws from a Markov matrix $Q(e_t|e_{t-1})$ for $t = 1, 2, 3, 4$.
- How can you implement this in Matlab?

Implementation

- Let's assume specifically that

$$\begin{aligned}
\bar{Q}(e^1) &= \bar{Q}(e^2) \\
&= 0.5 \\
Q(e^1|e^1) &= 0.9 \\
Q(e^2|e^1) &= 0.1 \\
Q(e^1|e^2) &= 0.1 \\
Q(e^2|e^2) &= 0.9
\end{aligned}$$

  meaning that the process is quite persistent.

- Let's assume $e^1 = 1$ and $e^2 = 2$ for simplicity.
- Let's use the uniform generator in Matlab.
- Generate the shocks for two households for five years.

Implementation

```
clear;clc;
rng(10)
Q_bar = [0.5; 0.5];
Q = [0.9, 0.1; 0.1, 0.9];
unif_draws = rand(5,2);
e_draws = zeros(5,2);
for i = 1:2
        if unif_draws(1,i) < Q_bar(1,1)
                e_draws(1,i) = 1;
        else
                e_draws(1,i) = 2;
        end
        for t = 2:5
                if (e_draws(t-1,i) == 1)
                        if unif_draws(t,i) < Q(1,1)
                                e_draws(t,i) = 1;
                        else
                                e_draws(t,i) = 2;
                        end
                else
                        if unif_draws(t,i) < Q(2,2)
                                e_draws(t,i) = 2;
                        else
                                e_draws(t,i) = 1;
                        end
                end
        end
end
```

Roadmap

Model Environment

- Here we'll just slightly augment the Aiyagari (1994) production economy model by adding productivity shocks.
- Study the model of Krusell & Smith (1998).

Model Environment

- The production technology now contains an aggregate productivity state, z_t

$$Y_t = F(z_t, K_t, L_t)$$

we'll just assume $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$ for $\alpha \in [0,1]$.

- Also assume that $z_t \in \{z^g, z^b\}$ where $z^b < z^g$.

Model Environment

- Households still face employment shocks that enter into their budget constraint of the form

$$k' + c = w\bar{e}e + (1 - \delta + r)k$$

where we'll now assume that e takes two values $e \in \{e^g, e^b\}$ where $e^b < e^g$.

Model Environment

- Households still face employment shocks that enter into their budget constraint of the form

$$k' + c = w\bar{e}e + (1 - \delta + r)k$$

  where we'll now assume that e takes two values $e \in \{e^g, e^b\}$ where $e^b < e^g$.

- Business cycles: we'd think that e is correlated with z.

Model Environment

- We assume that the shocks (e, Z) evolve jointly via a Markov chain of the form

$$Q(e', z'|e, z) = \Pr(e_{t+1} = e', z_{t+1} = z'|e_t = e, z_t = z)$$

- More specifically, we'd likely expect that it's easier to work a lot when the economy is booming and that remaining employed is harder when entering a recession.

- In the math, this would mean that

$$Q(e^g, z^b|e^b, z^b) < Q(e^g, z^g|e^b, z^g)$$
$$Q(e^g, z^b|e^g, z^b) < Q(e^g, z^g|e^g, z^g)$$

- The joint Markov matrix would have 16 entries in this 2 by 2 case.

Model Environment

- What states are relevant for the household's problem?
- The individual states $(k, e) \in \mathscr{K} \times \mathscr{E}$ affect their budget constraint directly.
- What aggregate states matter though?...

Model Environment

- Only current capital matters for the current wage and rental rate.
- What about for next period though?
- Is aggregate capital sufficient?

Model Environment

- No!
- The distribution of asset holdings is what matters.
- Why?

Model Environment

- Recall that a Bellman equation, (for a simple problem), is of the form

$$v(\vec{a}_t) = \max_{\vec{x}_t} \; u(a_t, x_t) + \beta \mathbb{E}_t[v(\vec{a}_{t+1})]$$

for state $\vec{a}_t$ and controls $\vec{x}_t$.

- The object $\mathbb{E}_t[v(\vec{a}_{t+1})]$ is not just $\mathbb{E}_t[u(a_{t+1}, x_{t+1})]$.
- It contains more information than just next period's period payoff.

Model Environment

- Object $\mathbb{E}_t[v(\vec{a}_{t+1})]$ is often referred to as the continuation value of the optimisation problem.
- It contains information pertaining to all future time periods.
- The household needs to know where K_t is headed going forward to properly make their own savings decisions.
- This fluctuates with the business cycles.

Model Environment

- E.g. the return to investment two periods from now depends on the stage of the business cycle, interacted with employment shocks, interacted with savings decisions.
- Alternatively: you can write-down two optimisation problems with the same current capital stock and business cycle stage, which give different value functions for households.
- The whole cross-section today and in the future is what matters.

Model Equilibrium

- Household's recursive problem is

$$v(a, s; z, \mu) = \max_{c, a'} \; u(c) + \beta \mathbb{E}_{s', z'}[v(a', s'; z', \mu')]$$

where

$$\begin{aligned} c + k' &= w(z, K)\bar{e}e + (1 - \delta + r(z, K))k \\ k &\geqslant 0 \\ K &= \int_{\mathscr{K} \times \mathscr{E}} k \; d\mu \\ \mu' &= G(z, \mu, z') \end{aligned}$$

where $G(z, \mu, z')$ is the law of motion for the aggregate state.

- $G(z, \mu, z')$ is an endogenous object.

Model Equilibrium

- The issue is that the value function now depends on μ. This is a distribution.
- Complication: the cross-section is an infinite dimensional object.

Model Equilibrium

- Firm problem is given by

$$\max_{L,K} \; zK^{\alpha}L^{1-\alpha} - w(z,K)L - r(z,K)K$$

giving the usual FOCs

$$\begin{aligned} w(z,K) &= (1-\alpha)z\left(\frac{K}{L}\right)^{\alpha} \\ r(z,K) &= \alpha z\left(\frac{K}{L}\right)^{\alpha-1} \end{aligned}$$

Model Equilibrium

- A recursive competitive equilibrium for this economy is a value function v, policy functions for the household a' and c, policies for the firm L and K, pricing functions r and w and a law of motion G such that the following conditions hold:
- Given the pricing functions, $r(z, K)$ and $w(z, K)$, the policy functions a' and c solve the household's problem and v is the associated value function,
- Given the pricing functions, the firms optimally hire labour and capital,
- The labour market and capital market clear

$$L = \bar{e} \int_{\mathscr{K} \times \mathscr{E}} e \ d\mu$$

$$K = \int_{\mathscr{K} \times \mathscr{E}} k \ d\mu$$

Model Equilibrium

- The goods market clears

$$\int_{\mathscr{K}\times\mathscr{E}} c(k,e;z,\mu)d\mu + \int_{\mathscr{K}\times\mathscr{E}} k'(k,e;z,\mu)d\mu = zK^{\alpha}L^{1-\alpha} + (1-\delta)K,$$

- The aggregate law of motion G is generated by the exogenous Markov chan and the policy functions

$$G(z,\mu,z') = \int_{\mathscr{K}\times\mathscr{E}} \tilde{Q}_{z,z'}(k,e)d\mu(k,e)$$

where $\tilde{Q}_{z,z'}(k,e)$ is a transition function from z to z' given by

$$\tilde{Q}_{z,z'}(k,e) = \sum_{e\in\mathscr{E}} \mathbb{1}_{k'(k,e;z,\mu)\in\mathscr{K}} Q(e',z'|e,z).$$

Model Equilibrium

- Our macro T.A. in the first year of Ph.D. classes once said

  Krusell and Smith (1998) is easy to talk about...but not so easy to compute... (Dempsey, 2014).

- Let's find out why...

Model Equilibrium

- Issue: μ_t is now a state.
- The Krusell & Smith (1998) algorithm relies on approximating this infinite-dimensional object with something that's finite.
- They solve for an approximate equilibrium.
- Remember back to statistics: any distribution can be represented by its entire (generally infinite in number) set of moments.

Model Equilibrium

- Denote $\bar{m}$ the M dimensional vector of the wealth distribution's (i.e. marginal of μ with respect to k) first M moments.
- We can then approximate the μ state with

$$\bar{m} = \{m^1, m^2, ..., m^M\}$$

where the moments are the mean, variance, skewness, kurtosis, etc.

- This moment vector will then also have a law of motion given by

$$\bar{m}' = G_M(z, \bar{m})$$

where notice now that z' has dropped-out since we're only interested in the wealth cross-section.

Model Equilibrium

- Agents in the model don't have full information about the distribution, but only these M moments.
- They can use them to approximate the distribution for their optimal decisions.

Model Equilibrium

- Their suggestion is to specify a law of motion of the form

$$\log((K')^1) = b_z^0 + b_z^1 \log(K^1) + b_z^2 \log(K^2) + \ldots + b_z^M \log(K^M)$$

where they find that setting $M = 1$ actually gives a pretty good approximation.

- Just use

$$\log((K^{'})^1) = b_z^0 + b_z^1 \log(K^1)$$

i.e. the mean of the distribution today maps into the mean of the distribution tomorrow linearly.

Model Equilibrium

- We can then solve for a partial information equilibrium where the household solves

$$v(k, e; z, K) = \max_{c,a'} \; u(c) + \beta \mathbb{E}_{e',z'}[v(k', e'; z', K')]$$

where

$$\begin{aligned} k' + c &= w(z, K)\bar{e}e + (1 - \delta + r(z, K))k \\ k' &\geqslant 0 \\ \log((K')^1) &= b_z^0 + b_z^1 \log(K^1), \end{aligned}$$

where we're now approximating the infinite-dimensional distribution with a single number.

- The last line is an autoregression from today's mean to tomorrow's.

Model Equilibrium

- This problem looks a bit like the RCE we we're studying in the representative agent framework.
- Recall the idea of consistency: where we wanted the law of motion to be consistent with the choices made by the agent.
- We need to do something similar here by choosing the coefficients $\{b_z^0, b_z^1\}$ appropriately.
- Why do we have z subscripts on the coefficients?
- Recall we had two values of $z \in \{z^g, z^b\}$.
- Therefore we have two pairs of these coefficients.

Model Equilibrium

- The Krusell & Smith (1998) algorithm is as follows
  - **(1)** Guess the coefficients $\vec{b}_z \equiv \{b_z^0, b_z^1\}$.
  - **(2)** Solve the household problem to get decision rules $k'(k, e; z, K; \vec{b}_z)$ and $c(k, e; z, K; \vec{b}_z)$.
  - **(3)** Simulate the economy for $I \in \mathbb{N}$ individuals for $T \in \mathbb{N}$ periods, (using random sequences of aggregate and individual shocks).
  - **(4)** Use the decision rules to generate sequences of wealth holdings, $\{\{k_t^i\}_{t=1}^T\}_{i=1}^I$ and find the mean capital stock for each period

$$K_t^1 = \frac{1}{I}\sum_{i=1}^{I} k_t^i$$

  - **(5)** After discarding the first $\hat{T}$ periods to remove dependence on the starting point, estimate the regression equation

$$\log(K_{t+1}^1) = \hat{b}_z^0 + \hat{b}_z^1 \log(K_t^1)$$

Model Equilibrium

(6) If the new estimates $\{\hat{b}_z^0, \hat{b}_z^1\} \approx \{b_z^0, b_z^1\}$ then stop. If the two pairs are equal for each $z \in \{z^g, z^b\}$ then we have consistency of the law of motion.

(7) Check how good the approximation is: can just use R^2 from the regression for this purpose. If it's a good fit, (KS, 1998 find it to be close to 99% R^2 using the first moment), then you're done. If not, try adding another moment and repeating; see if R^2 improves. If it doesn't change much, then stop. Otherwise keep adding moments until R^2 is satisfactory.

K-S (1998) Legacy

- Two contributions.

1. Methodological: how do we solve heterogeneous agent models with both idiosyncratic and aggregate uncertainty?
2. Actually show that the representative agent assumption gives a good approximation to this fancier setup.....which is....yeah....

Roadmap

Takeaways

- This area is usually referred to as quantitative macroeconomics.
- If you can do all the problem sets, you've got all the tools really.
- The firm dynamics course will hopefully re-enforce these techniques.
- What other exercises can you do?
- Calibrate something using SMM. Solve for transitions of models, both representative and heterogeneous agent.
- Or better still...

Takeaways

- Do research in the area!
- The harder part from here is coming-up with your research question.
- Me, Alessandro, Jake, Giammario all do a lot of this stuff.
- My part doesn't stop here: feel free to stop-by my office or email me anytime!