## Lecture 7: Overlapping Generations Model

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#### Roadmap

#### Introduction

2 Environment without Money

3 Equilibrium without Money

4 Environment with Money

5 Equilibrium with Money

#### 6 Summary

## Motivation

• In our models so far, we've had infinitely-lived representative households.



• What happens if we relax this assumption? Why?

## Preview of the Model

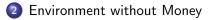
- We'll develop a model with "young" and "old" generations in each period.
- Each generation will live for two time periods and die.
- Start from t = 0 (with initial old and young) and time goes on forever.
- Receive an endowment of perishable consumption goods in each period.
- Endowment is larger when young than when old.
- Motive for consumption smoothing, in principle, gives motivation for trading between young and old.

## Preview of the Punchline

- In the absence of some tradable asset, the equilibrium of this model will not be Pareto optimal.
- Age structure of the model creates problems.
- Why would the young ever give any of their endowment to the old?
- Money can move us towards the efficient allocation.

## Roadmap

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## Setup

- Young and old each period.
- $N_t$  new people are born each period for t = 0, 1, ...
- Period t = 0 starts with an initial old of mass  $N_{-1}$ .
- At period t = 0, total number of people is  $N_{-1} + N_0$ .
- For general t, total people is  $N_{t-1} + N_t$ .

#### Setup

- Young are given endowment of 2 goods each period; old 1 good.
- Total endowment of consumption goods at t given by  $N_{t-1} + 2N_t$ .
- Assume time separable and concave utility function.
- Concavity generates motivation for consumption in both periods of life.
- Two types of households: those born from t = 0 onwards and the initial old.
- Households can hold one period bonds  $b_t$  that offer gross return  $R_t > 1$ .
- Assume for simplicity that  $N_t = N \ \forall t$ .

# Utility Function

Assume log utility

$$U(c_{1,t}, c_{2,t}) = \log(c_{1,t}) + \beta \log(c_{2,t})$$

where  $0 < \beta < 1$  is the discount factor and  $c_{j,t}$  for  $j \in \{1,2\}$  denotes consumption in the  $j^{th}$  period of life for a household born in period t.

#### Household Problem: t = 0 Onwards

• A household born in t = 0 or later solves the problem

$$\max_{\{c_{1,t},c_{2,t},b_t\}} \log(c_{1,t}) + \beta \log(c_{2,t})$$

subject to budget constraints in each period

$$c_{1,t} + b_t \le 2$$
$$c_{2,t} \le R_t b_t + 1$$

## Household Problem: Initial Old

• The initial old solve the problem

$$\max_{\{c_{2,-1}\}} \log(c_{2,-1})$$

subject to budget constraints in each period

$$c_{2,t} \leq 1$$

### Roadmap





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## Household Solution: t = 0 Onwards

• Substitute the budget constraints into the utility function to get the objective

$$\mathcal{L} = \log(2 - b_t) + \beta \log(R_t b_t + 1)$$

why do the two constraints bind?

FOC

$$\begin{split} \frac{\partial \mathcal{L}}{\partial b_t} &= 0 \Rightarrow -\frac{1}{2-b_t} + \frac{\beta R_t}{R_t b_t + 1} = 0\\ &\Rightarrow b_t = \frac{2R_t\beta - 1}{R_t[1+\beta]} \end{split}$$

## Household Solution: Initial Old

• Simply consume your endowment

 $c_{2,-1} = 1$ 

# Equilibrium Definition

- An equilibrium of the OLG model is defined such that
  - All agents optimise subject to budget constraints.
  - Prices all taken as given.
  - Bond market clears: assume that the sum of bond holdings across all households equals zero, (known as zero net supply).
    - Says that the savers and borrowers cancel each other out.
  - Goods market clears: sum of total consumption equals sum of total endowments.

## Bond Market Clearing

- See that since all the households born in a given period are identical, it follows that b<sub>t</sub> = 0 ∀t.
- No lending/borrowing takes place.
- The equilibrium R<sub>t</sub> consistent with this is such that

$$b_t = 0$$
  
 $\Rightarrow R_t = rac{1}{2eta}.$ 

What does this mean? It's the discounted marginal utility ratio at the endowment points.

# Autarky

- Equilibrium is autarky.
- We simply consume our endowments.
- No trade between generations as the old can't pay back the young next period!
- The current asset market structure doesn't allow one to save for old age, (which they wish to do since their endowment is higher when young by assumption).

- A Pareto improvement takes place when, through reallocation, we make one generation better-off without making another worse-off.
- In autarky, the initial old and future generations receive utilities of  $\log(1) = 0$  and  $\log(2) + \beta \log(1) = \log(2)$  respectively.
- A Pareto improvement would then take place if we could re-allocate resources such that initial old get utility above zero and future generations get at least log(2) without violating resource constraint.

- Resource constraint here is  $c_{1,t} + c_{2,t-1} = 3$ . Why?
- Consider re-distributing all 0.1 of the young's endowment each period to the old.
- Initial old get utility log(1.1) (strictly better off).
- All future generations get lifetime utility  $\log(1.9) + \beta \log(1.1)$ .

• This represents a Pareto improvement when

$$egin{aligned} \log(1.9)+eta\log(1.1)>\log(2)\ &\Rightarroweta>rac{\log(2)-\log(1.9)}{\log(1.1)}pprox0.54. \end{aligned}$$

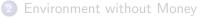
since this ensures the future generations aren't harmed.

- Improvement if all future generations are "patient enough".
- The equilibrium is inefficient.

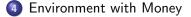
- How can we facilitate such a re-distribution from young to old in each period?
- There are several approaches, which involve government intervention.
- We'll focus here on using fiat money.

## Roadmap





3 Equilibrium without Money





#### 6 Summary

# Including Money

- Say that the central bank issues  $\mathcal{M}$  units of money at t = 0 amongst the initial old.
- Each initial old receives  $\frac{\mathcal{M}}{N}$  of cash.
- Assume that the money supply then stays constant forever.

# Including Money

- So far we've been thinking about prices in terms of money.
- E.g. *p* units of money buy one consumption good.
- Here we'll think about the opposite: 1 unit of money buys  $\phi_t$  units of consumption goods, (we'll allow it to be time-dependent here).
- Just the inverse of p.

#### Household Problem: t = 0 Onwards

• A household born in t = 0 or later solves the problem

$$\max_{\{c_{1,t},c_{2,t},m_t\}} \log(c_{1,t}) + \beta \log(c_{2,t})$$

where  $m_t$  is the amount of money they take into the next period. They solve the problem subject to budget constraints in each period

$$c_{1,t} + \phi_t m_t \le 2$$
$$c_{2,t} \le \phi_{t+1} m_t + 1$$

where  $\phi_t$  is the amount of consumption goods that a unit of money can buy in t (the inverse of the goods price).

## Household Problem: Initial Old

• The initial old solve the problem

$$\max_{\{c_{2,-1}\}} \log(c_{2,-1})$$

subject to budget constraints

$$c_{2,-1} \leq 1 + \phi_0 \frac{\mathcal{M}}{N}$$

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### Household Solution: t = 0 Onwards

• Substitute the budget constraints into the utility function to get the objective

$$\mathcal{L} = \log(2 - \phi_t m_t) + \beta \log(\phi_{t+1} m_t + 1)$$

FOC

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial m_t} &= 0 \Rightarrow -\frac{\phi_t}{2 - \phi_t m_t} + \frac{\beta \phi_{t+1}}{\phi_{t+1} m_t + 1} = 0 \\ &\Rightarrow m_t = \frac{2\phi_{t+1}\beta - \phi_t}{[1 + \beta]\phi_t \phi_{t+1}} \end{aligned}$$
(1)

### Household Solution: Initial Old

#### • Simply consume your endowment

$$c_{2,-1} = 1 + \phi_0 \frac{\mathcal{M}}{N}$$

# Equilibrium Definition

- A equilibrium of the OLG model with money is defined such that
  - All agents optimise subject to budget constraints.
  - Prices all taken as given.
  - $\bullet\,$  Money market clears: aggregate money demand equals  $\mathcal{M}.$
  - Goods market clears: sum of total consumption equals sum of total endowments.

#### Money Market Clearing

- The net supply of money is  $\mathcal{M}$ . How does this differ from bonds in the earlier version of the model? What does this mean?
- Market clearing requires that

$$\mathcal{M} = m_t N \tag{2}$$
$$\Rightarrow m_t = \frac{\mathcal{M}}{N}$$

Using (1)

$$\frac{2\phi_{t+1}\beta - \phi_t}{[1+\beta]\phi_t\phi_{t+1}} = \frac{\mathcal{M}}{\mathcal{N}}$$

$$\Rightarrow \phi_t = \frac{2\phi_{t+1}\beta\frac{\mathcal{N}}{\mathcal{M}}}{(1+\beta)\phi_{t+1} + \frac{\mathcal{N}}{\mathcal{M}}}$$
(3)

which is increasing in  $\phi_{t+1}$ , decreasing in the fixed supply of money and increasing in the population size.

## Value of Money

- Let's look for an equilibrium, which is stationary.
- This means that for all future generations,  $c_{1,t} = \bar{c}_1$  and  $c_{2,t} = \bar{c}_2$ .
- By (2), it must be that  $m_t$  is also constant.
- Follows from the period budget constraints that  $\phi_t$  is also constant at some price  $\bar{\phi}$ .

# Value of Money

• Can then solve equation (3) for  $ar{\phi}$ 

$$\bar{\phi} = \left(\frac{N}{\mathcal{M}}\right) \frac{[2\beta - 1]}{1 + \beta}$$

which is positive if  $\beta > 1/2$ : if agents are sufficiently "patient".

• Given this, it follows that the value of money holdings are given by

$$ar{\phi}ar{m}=rac{2eta-1}{1+eta}$$

### Money and Welfare

• The values of consumption for future generations are

$$ar{c}_1 = rac{3}{1+eta} \ ar{c}_2 = rac{3eta}{1+eta}$$

which yields welfare of

$$\log\left(\frac{3}{1+\beta}\right) + \beta \log\left(\frac{3\beta}{1+\beta}\right)$$

which is always weakly above the autarky welfare (exercise: show).

## Money and Welfare

The consumption of initial old is then given by

$$egin{aligned} c_{2,-1} &= 1 + \left(rac{N}{\mathcal{M}}
ight)rac{\left[2eta-1
ight]}{1+eta}rac{\mathcal{M}}{N} \ &= 1 + rac{2eta-1}{1+eta} \end{aligned}$$

which is greater than unity if money has value.

- The monetary equilibrium yields a Pareto improvement over the autarkic equilibrium!
- Why?

## Money and Welfare

- Money allows people to save!
- Big emphasis on the store of value function here.
- Since the endowments of consumption goods were perishable, there was no store of value in the autarky equilibrium.
- Money makes us all better off.

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#### Takeaways

- OLG model is a workhorse of macro.
- Timing structure for generations can lead to inefficiency.
- Money in this model can yield Pareto improvements.
- In fact, the monetary equilibrium we found **is** the Pareto optimal allocation (exercise).