

Lecture 7: Overlapping Generations Model

Adam Hal Spencer

The University of Nottingham

Advanced Monetary Economics 2020

Roadmap

- 1 Introduction
- 2 Environment without Money
- 3 Equilibrium without Money
- 4 Environment with Money
- 5 Equilibrium with Money
- 6 Summary

Motivation

- In our models so far, we've had infinitely-lived representative households.



- What happens if we relax this assumption? Why?

Preview of the Model

- We'll develop a model with “young” and “old” generations in each period.
- Each generation will live for two time periods and die.
- Start from $t = 0$ (with initial old and young) and time goes on forever.
- Receive an endowment of **perishable** consumption goods in each period.
- Endowment is larger when young than when old.
- Motive for consumption smoothing, in principle, gives motivation for trading between young and old.

Preview of the Punchline

- In the absence of some tradable asset, the equilibrium of this model will not be Pareto optimal.
- Age structure of the model creates problems.
- Why would the young ever give any of their endowment to the old?
- Money can move us towards the efficient allocation.

Roadmap

- 1 Introduction
- 2 Environment without Money**
- 3 Equilibrium without Money
- 4 Environment with Money
- 5 Equilibrium with Money
- 6 Summary

Setup

- Young and old each period.
- N_t new people are born each period for $t = 0, 1, \dots$
- Period $t = 0$ starts with an initial old of mass N_{-1} .
- At period $t = 0$, total number of people is $N_{-1} + N_0$.
- For general t , total people is $N_{t-1} + N_t$.

Setup

- Young are given endowment of 2 goods each period; old 1 good.
- Total endowment of consumption goods at t given by $N_{t-1} + 2N_t$.
- Assume time separable and concave utility function.
- Concavity generates motivation for consumption in **both periods** of life.
- Two types of households: those born from $t = 0$ onwards and the initial old.
- Households can hold one period bonds b_t that offer gross return $R_t > 1$.
- Assume for simplicity that $N_t = N \forall t$.

Utility Function

- Assume log utility

$$U(c_{1,t}, c_{2,t}) = \log(c_{1,t}) + \beta \log(c_{2,t})$$

where $0 < \beta < 1$ is the discount factor and $c_{j,t}$ for $j \in \{1, 2\}$ denotes consumption in the j^{th} period of life for a household born in period t .

Household Problem: $t = 0$ Onwards

- A household born in $t = 0$ or later solves the problem

$$\max_{\{c_{1,t}, c_{2,t}, b_t\}} \log(c_{1,t}) + \beta \log(c_{2,t})$$

subject to budget constraints in each period

$$c_{1,t} + b_t \leq 2$$

$$c_{2,t} \leq R_t b_t + 1$$

Household Problem: Initial Old

- The initial old solve the problem

$$\max_{\{c_{2,-1}\}} \log(c_{2,-1})$$

subject to budget constraints in each period

$$c_{2,t} \leq 1$$

Roadmap

- 1 Introduction
- 2 Environment without Money
- 3 Equilibrium without Money**
- 4 Environment with Money
- 5 Equilibrium with Money
- 6 Summary

Household Solution: $t = 0$ Onwards

- Substitute the budget constraints into the utility function to get the objective

$$\mathcal{L} = \log(2 - b_t) + \beta \log(R_t b_t + 1)$$

why do the two constraints bind?

- FOC

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial b_t} = 0 &\Rightarrow -\frac{1}{2 - b_t} + \frac{\beta R_t}{R_t b_t + 1} = 0 \\ &\Rightarrow b_t = \frac{2R_t\beta - 1}{R_t[1 + \beta]}\end{aligned}$$

Household Solution: Initial Old

- Simply consume your endowment

$$c_{2,-1} = 1$$

Equilibrium Definition

- An equilibrium of the OLG model is defined such that
 - All agents optimise subject to budget constraints.
 - Prices all taken as given.
 - Bond market clears: assume that the sum of bond holdings across all households equals zero, (known as zero net supply).
 - Says that the savers and borrowers cancel each other out.
 - Goods market clears: sum of total consumption equals sum of total endowments.

Bond Market Clearing

- See that since all the households born in a given period are identical, it follows that $b_t = 0 \forall t$.
- No lending/borrowing takes place.
- The equilibrium R_t consistent with this is such that

$$b_t = 0$$
$$\Rightarrow R_t = \frac{1}{2\beta}.$$

What does this mean? It's the discounted marginal utility ratio at the endowment points.

Autarky

- Equilibrium is **autarky**.
- We simply consume our endowments.
- No trade between generations as the old can't pay back the young next period!
- The current asset market structure doesn't allow one to save for old age, (which they wish to do since their endowment is higher when young by assumption).

Pareto Improvements

- A **Pareto improvement** takes place when, through reallocation, we make one generation better-off without making another worse-off.
- In autarky, the initial old and future generations receive utilities of $\log(1) = 0$ and $\log(2) + \beta \log(1) = \log(2)$ respectively.
- A Pareto improvement would then take place if we could re-allocate resources such that initial old get utility **above zero** and future generations get at least $\log(2)$ without violating resource constraint.

Pareto Improvements

- Resource constraint here is $c_{1,t} + c_{2,t-1} = 3$. Why?
- Consider re-distributing all 0.1 of the young's endowment each period to the old.
- Initial old get utility $\log(1.1)$ (strictly better off).
- All future generations get lifetime utility $\log(1.9) + \beta \log(1.1)$.

Pareto Improvements

- This represents a Pareto improvement when

$$\begin{aligned}\log(1.9) + \beta \log(1.1) &> \log(2) \\ \Rightarrow \beta &> \frac{\log(2) - \log(1.9)}{\log(1.1)} \approx 0.54.\end{aligned}$$

since this ensures the future generations aren't harmed.

- Improvement if all future generations are “patient enough”.
- The equilibrium is inefficient.

Pareto Improvements

- How can we facilitate such a re-distribution from young to old in each period?
- There are several approaches, which involve government intervention.
- We'll focus here on using fiat money.

Roadmap

- 1 Introduction
- 2 Environment without Money
- 3 Equilibrium without Money
- 4 Environment with Money**
- 5 Equilibrium with Money
- 6 Summary

Including Money

- Say that the central bank issues \mathcal{M} units of money at $t = 0$ amongst the initial old.
- Each initial old receives $\frac{\mathcal{M}}{N}$ of cash.
- Assume that the money supply then stays constant forever.

Including Money

- So far we've been thinking about prices in terms of money.
- E.g. p units of money buy one consumption good.
- Here we'll think about the opposite: 1 unit of money buys ϕ_t units of consumption goods, (we'll allow it to be time-dependent here).
- Just the inverse of p .

Household Problem: $t = 0$ Onwards

- A household born in $t = 0$ or later solves the problem

$$\max_{\{c_{1,t}, c_{2,t}, m_t\}} \log(c_{1,t}) + \beta \log(c_{2,t})$$

where m_t is the amount of money they take into the next period. They solve the problem subject to budget constraints in each period

$$\begin{aligned} c_{1,t} + \phi_t m_t &\leq 2 \\ c_{2,t} &\leq \phi_{t+1} m_t + 1 \end{aligned}$$

where ϕ_t is the amount of consumption goods that a unit of money can buy in t (the inverse of the goods price).

Household Problem: Initial Old

- The initial old solve the problem

$$\max_{\{c_{2,-1}\}} \log(c_{2,-1})$$

subject to budget constraints

$$c_{2,-1} \leq 1 + \phi_0 \frac{\mathcal{M}}{N}$$

Roadmap

- 1 Introduction
- 2 Environment without Money
- 3 Equilibrium without Money
- 4 Environment with Money
- 5 Equilibrium with Money**
- 6 Summary

Household Solution: $t = 0$ Onwards

- Substitute the budget constraints into the utility function to get the objective

$$\mathcal{L} = \log(2 - \phi_t m_t) + \beta \log(\phi_{t+1} m_t + 1)$$

- FOC

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial m_t} = 0 &\Rightarrow -\frac{\phi_t}{2 - \phi_t m_t} + \frac{\beta \phi_{t+1}}{\phi_{t+1} m_t + 1} = 0 & (1) \\ &\Rightarrow m_t = \frac{2\phi_{t+1}\beta - \phi_t}{[1 + \beta]\phi_t\phi_{t+1}} \end{aligned}$$

Household Solution: Initial Old

- Simply consume your endowment

$$c_{2,-1} = 1 + \phi_0 \frac{\mathcal{M}}{N}$$

Equilibrium Definition

- A equilibrium of the OLG model with money is defined such that
 - All agents optimise subject to budget constraints.
 - Prices all taken as given.
 - Money market clears: aggregate money demand equals \mathcal{M} .
 - Goods market clears: sum of total consumption equals sum of total endowments.

Money Market Clearing

- The net supply of money is \mathcal{M} . How does this differ from bonds in the earlier version of the model? What does this mean?
- Market clearing requires that

$$\begin{aligned}\mathcal{M} &= m_t N & (2) \\ \Rightarrow m_t &= \frac{\mathcal{M}}{N}\end{aligned}$$

- Using (1)

$$\begin{aligned}\frac{2\phi_{t+1}\beta - \phi_t}{[1 + \beta]\phi_t\phi_{t+1}} &= \frac{\mathcal{M}}{N} & (3) \\ \Rightarrow \phi_t &= \frac{2\phi_{t+1}\beta\frac{N}{\mathcal{M}}}{(1 + \beta)\phi_{t+1} + \frac{N}{\mathcal{M}}}\end{aligned}$$

which is **increasing** in ϕ_{t+1} , decreasing in the fixed supply of money and increasing in the population size.

Value of Money

- Let's look for an equilibrium, which is **stationary**.
- This means that for all future generations, $c_{1,t} = \bar{c}_1$ and $c_{2,t} = \bar{c}_2$.
- By (2), it must be that m_t is also constant.
- Follows from the period budget constraints that ϕ_t is also constant at some price $\bar{\phi}$.

Value of Money

- Can then solve equation (3) for $\bar{\phi}$

$$\bar{\phi} = \left(\frac{N}{\mathcal{M}} \right) \frac{[2\beta - 1]}{1 + \beta}$$

which is positive if $\beta > 1/2$: if agents are sufficiently “patient”.

- Given this, it follows that the value of money holdings are given by

$$\bar{\phi} \bar{m} = \frac{2\beta - 1}{1 + \beta}$$

Money and Welfare

- The values of consumption for future generations are

$$\bar{c}_1 = \frac{3}{1 + \beta}$$
$$\bar{c}_2 = \frac{3\beta}{1 + \beta}$$

which yields welfare of

$$\log\left(\frac{3}{1 + \beta}\right) + \beta \log\left(\frac{3\beta}{1 + \beta}\right)$$

which is always weakly above the autarky welfare (exercise: show).

Money and Welfare

- The consumption of initial old is then given by

$$\begin{aligned}c_{2,-1} &= 1 + \left(\frac{N}{\mathcal{M}}\right) \frac{[2\beta - 1] \mathcal{M}}{1 + \beta} \frac{1}{N} \\ &= 1 + \frac{2\beta - 1}{1 + \beta}\end{aligned}$$

which is greater than unity if money has value.

- The monetary equilibrium yields a **Pareto improvement** over the autarkic equilibrium!
- Why?

Money and Welfare

- Money allows people to save!
- Big emphasis on the **store of value** function here.
- Since the endowments of consumption goods were perishable, there was no store of value in the autarky equilibrium.
- Money makes us all better off.

Roadmap

- 1 Introduction
- 2 Environment without Money
- 3 Equilibrium without Money
- 4 Environment with Money
- 5 Equilibrium with Money
- 6 Summary**

Takeaways

- OLG model is a workhorse of macro.
- Timing structure for generations can lead to inefficiency.
- Money in this model can yield Pareto improvements.
- In fact, the monetary equilibrium we found **is** the Pareto optimal allocation (exercise).